DATASCI 385 Quantitative Finance

Lecture 9: Markowitz model

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Suggested reading: Investments Ch 7

EMORY

Minimum variance portfolio

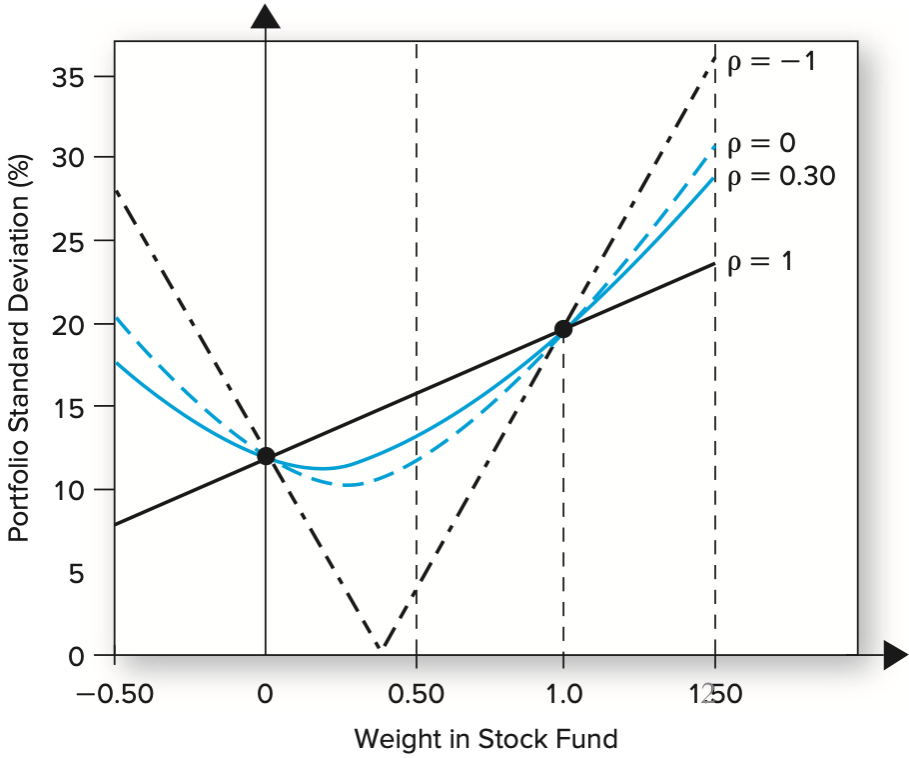
- Minimum-variance portfolio: Portfolio with w_D^* and w_E^* such that the corresponding σ_P^{*2} is the smallest among all possible choices of w_D and w_E

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2\rho_{DE} w_D w_E \sigma_D \sigma_E$$

- The optimal w_D^* that minimizes σ_P^2 is

$$w_D^* = \frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)}$$

	Debt	Equity
Expected return, $E(r)$	8%	13%
Standard deviation, σ	12%	20%



Maximum utility portfolio

- For the utility $U = E(r) - \frac{1}{2}A\sigma^2$ with the risk aversion parameter A, the optimal investment proportions in the two funds that maximizes U are

$$w_D^* = \frac{E(r_D) - E(r_E) + A(\sigma_E^2 - Cov(r_D, r_E))}{A(\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E))}$$

$$w_E^* = 1 - w_D^*$$

  - w_D^* increases with $E(r_D)$
  - w_D^* decreases with $E(r_E)$
  - w_D^* increases with A
    - $w_D^* = \frac{(E(r_D)-E(r_E))/A+(\sigma_E^2-Cov(r_D,r_E))}{(\sigma_D^2+\sigma_E^2-2Cov(r_D,r_E))} = \frac{\frac{-a}{A}+b}{c}$, where $a = E(r_E) - E(r_E) > 0$
    - $\frac{dw_D^*}{dA} = \frac{a}{cA^2} > 0$

Maximum Sharpe ratio portfolio

- The optimal portfolio weights that maximize Sharpe ratio is

$$w_D^* = \frac{E(R_D)\sigma_E^2 - E(R_E)Cov(R_D,R_E)}{E(R_D)\sigma_E^2 + E(R_E)\sigma_D^2 - [E(R_D)+E(R_E)]Cov(R_D,R_E)}$$

$$w_E^* = 1 - w_D^*$$

  - $R_D = r_D - r_f$ and $R_E = r_E - r_f$ are excess returns

Comparative statics

- The Sharpe ratio of the debt portfolio is

$$S_D = \frac{8-5}{12} = 0.25$$

- The Sharpe ratio of the equity portfolio is

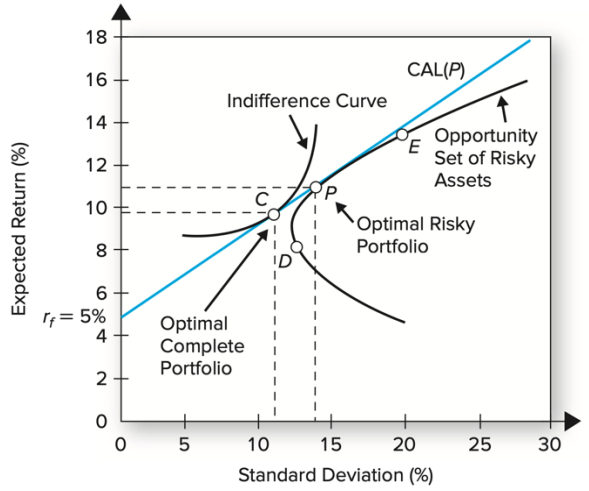
$$S_E = \frac{13-5}{20} = 0.4$$

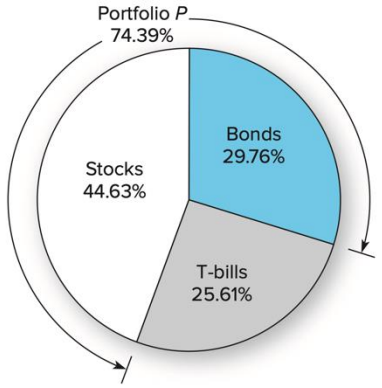
	Debt	Equity
Expected return, $E(r)$	8%	13%
Standard deviation, σ	12%	20%
Covariance, $\text{Cov}(r_D, r_E)$	72	
Correlation coefficient, ρ_{DE}	0.30	

- The Sharpe ratio of both debt and equity portfolios are lower than that of the *optimal risky portfolio* whose Sharpe ratio is .42
- Implication: Use the *optimal risky portfolio* to construct *optimal complete portfolio*

Optimal complete portfolio

1. Specify the return characteristics of all securities: expected returns, variances, covariances
2. Asset allocation decision to construct optimal risky portfolio P (same for all investors)
   - Solve the weight of each asset to maximize the Sharpe ratio of P
3. Capital allocation between risky portfolio P and risk-free assets (e.g., T-bills) to maximize an investor's utility (vary with investors)
   - Solve the weight y in P and weight $1 - y$ in risk-free assets to maximize utility score ($U = E(r) - \frac{1}{2}A\sigma^2$)





Markowitz model

- Markowitz model: portfolio optimization model, also called mean-variance model
  - A mathematical framework for assembling a portfolio of assets such that the expected return is maximized for a given level of risk

- Asset allocation among n assets: weight of asset i is w_i

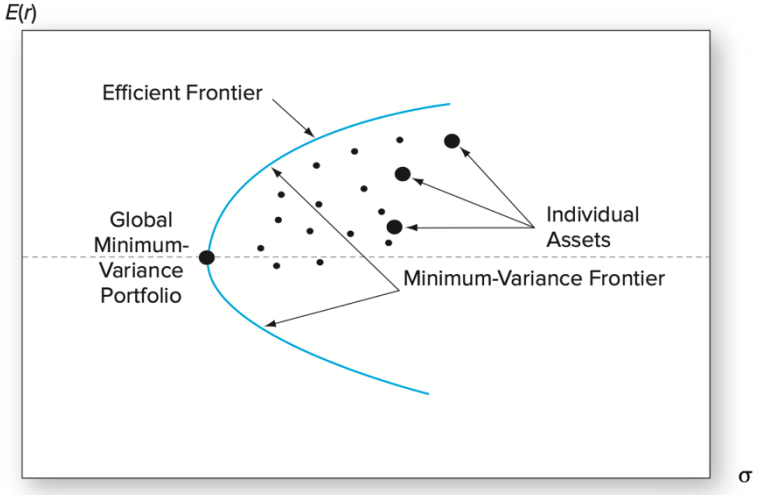
$$E(r_P) = \sum_{i=1}^{n} w_i E(r_i)$$

$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(r_i, r_j)$$

  - Portfolio manager has an estimate of $E(r_i)$ for all i and an estimate of $Cov(r_i, r_j)$ for all i and j

Minimum variance frontier

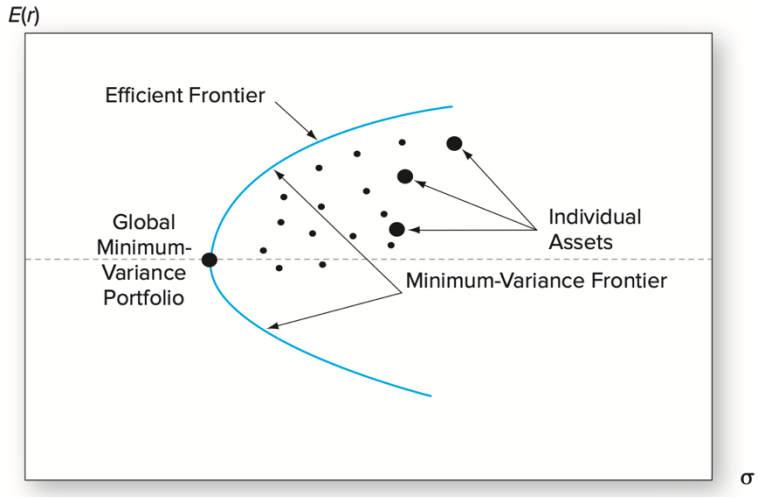
- Minimum variance frontier: A graph of the lowest possible variance that can be attained for a given portfolio expected return
- All the individual assets lie to the right inside the frontier (i.e., return is lower or risk is higher)
  - Short sales are allowed in the construction of risky portfolios
  - Portfolio with a single asset is inefficient
- Diversification can increase expected returns and lower standard deviations



Efficient frontier

- Efficient frontier is the portion of the minimum-variance frontier that lies above the global minimum-variance portfolio

- The bottom part of the minimum-variance frontier is inefficient
  - There is always a portfolio with the same standard deviation and a greater expected return positioned directly above it



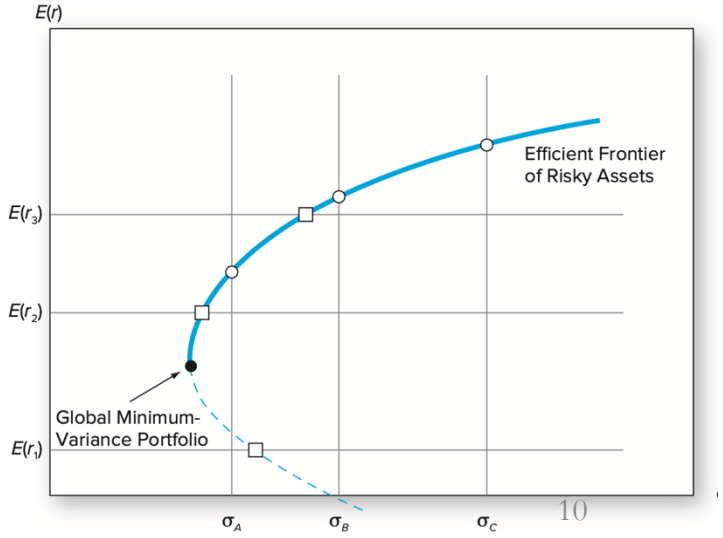
Two equivalent approaches to construct efficient frontier

- Approach 1: Minimize variance for any target expected return μ (e.g., points marked by squares)

$$\min_{w} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(r_i, r_j)$$
$$s.t. \sum_{i=1}^{n} w_i E(r_i) = \mu$$
$$\sum_{i=1}^{n} w_i = 1$$

- Approach 2: Maximize expected return for any target risk level σ^2 (e.g., points marked by circles)

$$\max_{w} \sum_{i=1}^{n} w_i E(r_i)$$
$$s.t. \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(r_i, r_j) = \sigma^2$$
$$\sum_{i=1}^{n} w_i = 1$$



Efficient frontier with constraints

- In the optimization problem

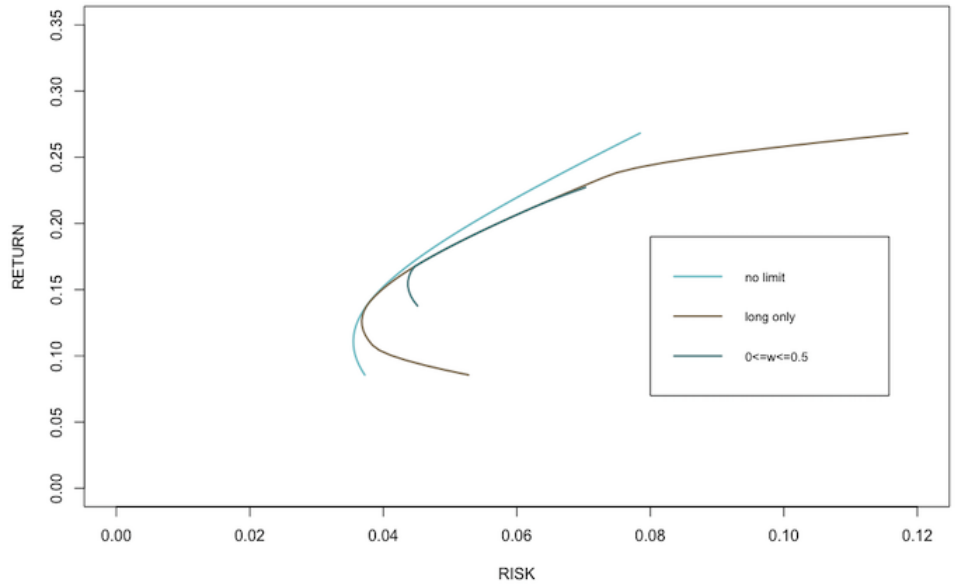
$$\min_{w} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(r_i, r_j)$$
$$s.t. \sum_{i=1}^{n} w_i E(r_i) = \mu$$
$$\sum_{i=1}^{n} w_i = 1$$

add additional constraints, e.g.,

  - Short-sell restrictions (non-negative weights)

$$w_i \geq 0$$

  - Weight restrictions
    - $w_i \leq \bar{w}$



- As we add more constraints, the frontier naturally moves more "to the **right**", i.e., towards higher risk and lower return

Markowitz model with risk-free assets

- Identify the point with the highest Sharpe ratio on the efficient frontier. This portfolio is denoted as P (shared across all investors)
- Construct the CAL that connects the risk-free asset with P
- The CAL is tangent to the efficient frontier
- For individual investors, choose the optimal complete portfolio with y in P and $1 - y$ in the risk-free asset to maximize $U = E[r_C] - \frac{1}{2} A\sigma_C^2$

