DATASCI 385 Quantitative Finance

Lecture 8: Efficient diversification

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Suggested reading: Investments Ch 7

EMORY

Two risky portfolios

- Two risky portfolios: a bond portfolio D specializing in long-term debt securities and a stock portfolio E specializing in equity securities
- Asset allocation decision of portfolio P: w_D in the bond portfolio and $w_E = 1 - w_D$ in the stock portfolio
  - Rate of return on this portfolio, r_P

$$r_P = w_D r_D + w_E r_E$$

  - Expected return on this portfolio

$$E(r_P) = w_D E(r_D) + w_E E(r_E)$$

  - Variance of this portfolio

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2 w_D w_E Cov(r_D, r_E) \qquad Cov(r_D, r_E) = \rho_{DE} \sigma_D \sigma_E$$

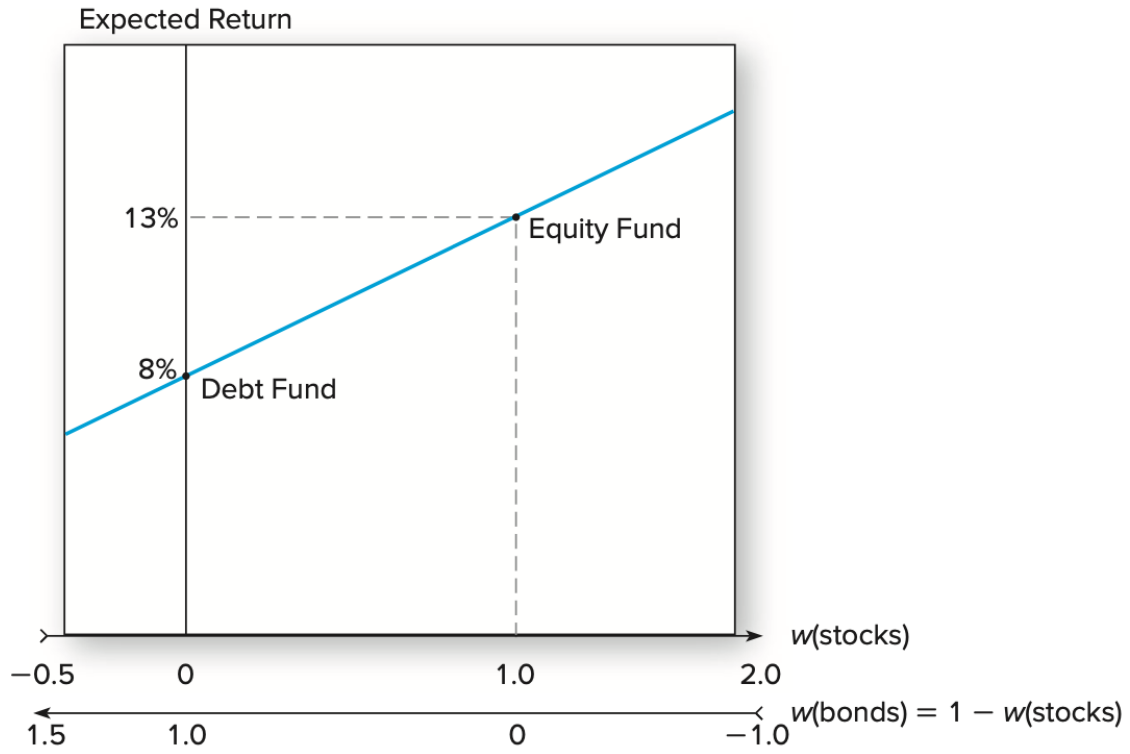
ρ_{DE} correlation coefficient of D and E

	Debt		Equity
Expected return, $E(r)$	8%		13%
Standard deviation, σ	12%		20%
Covariance, Cov(r_D, r_E)		72	
Correlation coefficient, ρ_{DE}		0.30	

Portfolio expected return as a function of w_E

- Expected return is linear in w_E

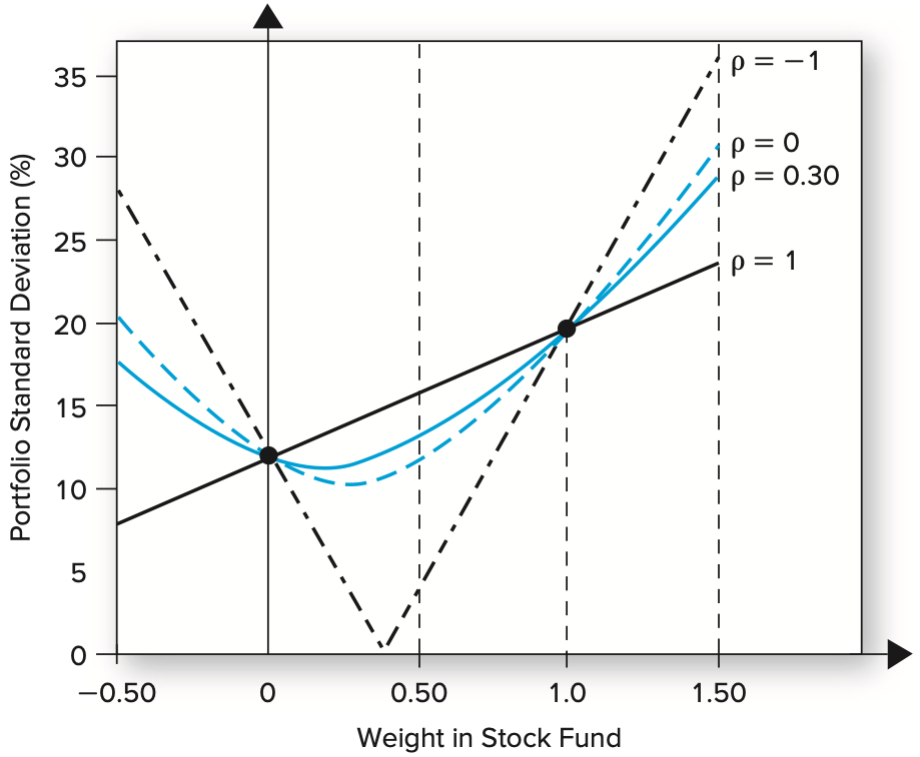
$$E(r_P) = w_D E(r_D) + w_E E(r_E) = (1 - w_E)E(r_D) + w_E E(r_E)$$
$$= E(r_D) + w_E\big(E(r_E) - E(r_D)\big)$$



Portfolio standard deviation as a function of w_E

- Variance of portfolio P is affected by the covariance of two portfolios

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2 w_D w_E \rho_{DE} \sigma_D \sigma_E$$



Covariance between two risky portfolios

- Variance of portfolio P is affected by the covariance of two portfolios

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2\rho_{DE} w_D w_E \sigma_D \sigma_E$$

  - Portfolio variance increases with ρ_{DE}

  - Case 1: $\rho_{DE} = 1$
  - D and E are perfectly positively correlated and the standard deviation of portfolio P is the largest

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2 w_D w_E \sigma_D \sigma_E = (w_D \sigma_D + w_E \sigma_E)^2$$
$$\Rightarrow \sigma_P = w_D \sigma_D + w_E \sigma_E$$

  - Standard deviation σ_P is linear in w_E and σ_E (also linear in w_D and σ_D)

Covariance between two risky portfolios

- Variance of portfolio P is affected by the covariance of two portfolios

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2\rho_{DE} w_D w_E \sigma_D \sigma_E$$

  - Case 2: $-1 < \rho_{DE} < 1$

$$\sigma_P < w_D \sigma_D + w_E \sigma_E$$

  - *Portfolios of less than perfectly correlated assets always offer some degree of diversification benefit*. The lower the correlation between the assets, the greater the gain in efficiency
  - If $\rho_{DE} < 0$, then D is a hedge portfolio of E. In this case, diversification is particularly effective in reducing total risk (while not affecting total return)

Covariance between two risky portfolios

- Variance of portfolio P is affected by the covariance of two portfolios

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2\rho_{DE} w_D w_E \sigma_D \sigma_E$$

  - Case 3: $\rho_{DE} = -1$
  - D and E are perfectly negatively correlated and

$$\sigma_P^2 = (w_D \sigma_D - w_E \sigma_E)^2$$
$$\Rightarrow \sigma_P = |w_D \sigma_D - w_E \sigma_E|$$

  - Standard deviation σ_P is the smallest

Perfectly hedging position

- A perfectly hedging position has zero risk ($\sigma_P = 0$)
- can be obtained by solving

$$w_D\sigma_D - w_E\sigma_E = 0$$

$$\Rightarrow w_D\sigma_D - (1 - w_D)\sigma_E = w_D(\sigma_D + \sigma_E) - \sigma_E = 0$$

- The solution is

$$w_D = \frac{\sigma_E}{\sigma_D + \sigma_E} \qquad w_E = \frac{\sigma_D}{\sigma_D + \sigma_E} = 1 - w_D$$

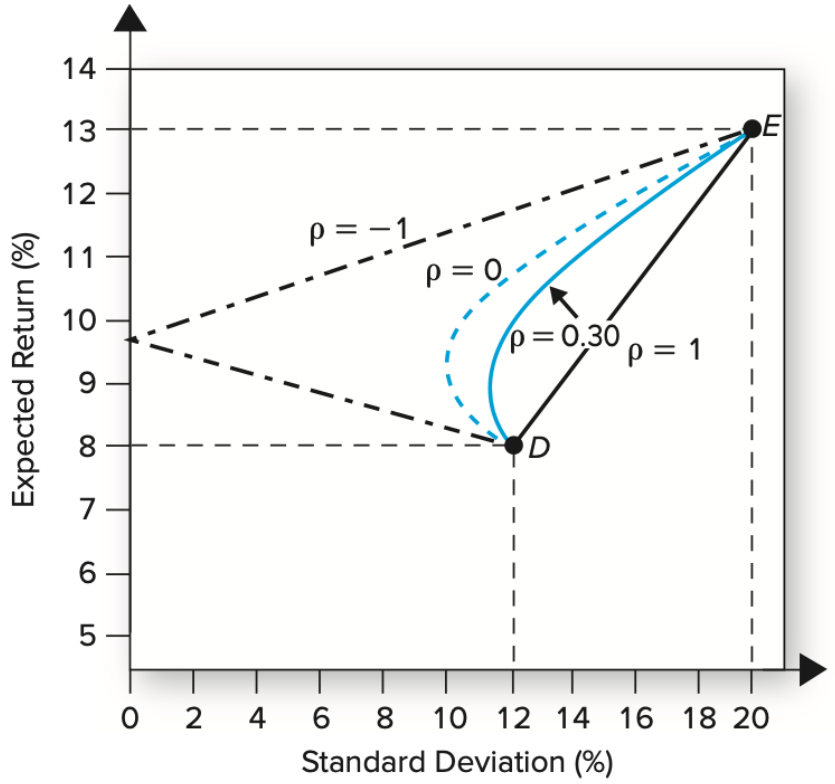
  - w_D increases with σ_E, decreases with σ_D

Portfolio opportunity set

- Portfolio opportunity set: all combinations of portfolio expected return and standard deviation that can be constructed from the two available assets/portfolios

- The lower the correlation, the greater the potential benefit from diversification

- When $\rho_{DE} = -1$, there is a perfect hedging opportunity with zero standard deviation



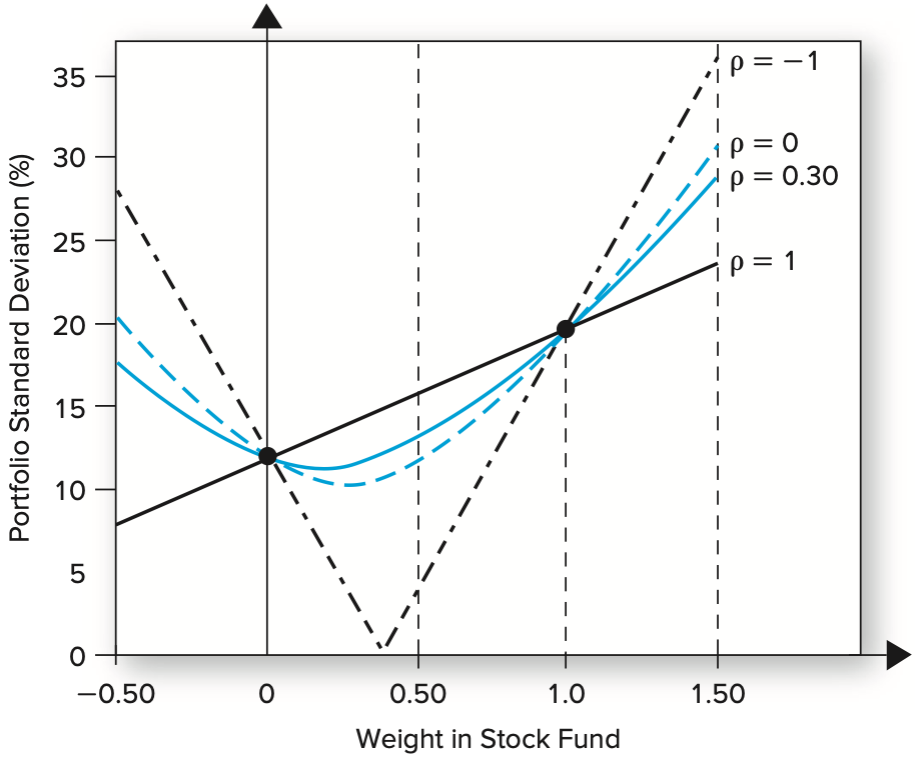
Minimum variance portfolio

- Minimum-variance portfolio: Portfolio with w_D^* and w_E^* such that the corresponding σ_P^{*2} is the smallest among all possible choices of w_D and w_E

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2\rho_{DE} w_D w_E \sigma_D \sigma_E$$

- If $\rho_{DE} = -1$, then
  - $w_D^* = \frac{\sigma_E}{\sigma_D + \sigma_E} = \frac{20}{12+20} = .625$
  - $w_E^* = 1 - .625 = .375$

	Debt	Equity
Expected return, $E(r)$	8%	13%
Standard deviation, σ	12%	20%



Minimum-variance portfolio for general ρ_{DE}

- Based on the variance formula

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2w_D w_E Cov(r_D, r_E)$$

- Replace w_E by $1 - w_D$ and take the derivative with respect to w_D. The optimal w_D^* that minimizes σ_P^2 is

$$w_D^* = \frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)}$$

Exercise: Derivation of optimal w_D^* to minimize variance

- Write σ_P^2 in terms of w_D only (using $w_D = 1 - w_E$)
- Take the derivative of σ_P^2 with respect to w_D and set it to zero

$$\frac{d\sigma_P^2}{dw_D} = 0$$

- Solve w_D from the first order condition

Minimum-variance portfolio with $\rho_{DE} = -1$

- When $\rho_{DE} = -1$

$$w_D^* = \frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)}$$

$$= \frac{\sigma_E^2 + \sigma_D \sigma_E}{\sigma_D^2 + \sigma_E^2 + 2\sigma_D \sigma_E}$$

$$= \frac{\sigma_E(\sigma_E + \sigma_D)}{(\sigma_E + \sigma_D)^2} = \frac{\sigma_E}{\sigma_E + \sigma_D}$$

- The same weight as the perfectly hedging position

Question

- The weight of in the minimum variance portfolio is

$$w_D^* = \frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)}$$

- Question: What is the weight of D and E when $\rho_{DE} = .3$? What if $\rho_{DE} = 0$?

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Expected return, $E(r)$	8%	13%
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Choosing a utility maximizing portfolio

- For the utility $U = E(r) - \frac{1}{2}A\sigma^2$ with the risk aversion parameter A, the optimal investment proportions in the two funds are

$$w_D^* = \frac{E(r_D) - E(r_E) + A(\sigma_E^2 - Cov(r_D, r_E))}{A(\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E))}$$

$$w_E^* = 1 - w_D^*$$

  - w_D^* increases with $E(r_D)$, decreases with $E(r_E)$
  - *Question: Does w_D^* increase or decrease with A?*

Exercise: Derivation of optimal w_D^* to maximize utility

- Write U in terms of w_D only (using $w_D = 1 - w_E$)
- Take the derivative of U with respect to w_D and set it to zero

$$\frac{dU}{dw_D} = 0$$

- Solve w_D from the first order condition

Choosing a portfolio based on Sharpe ratio

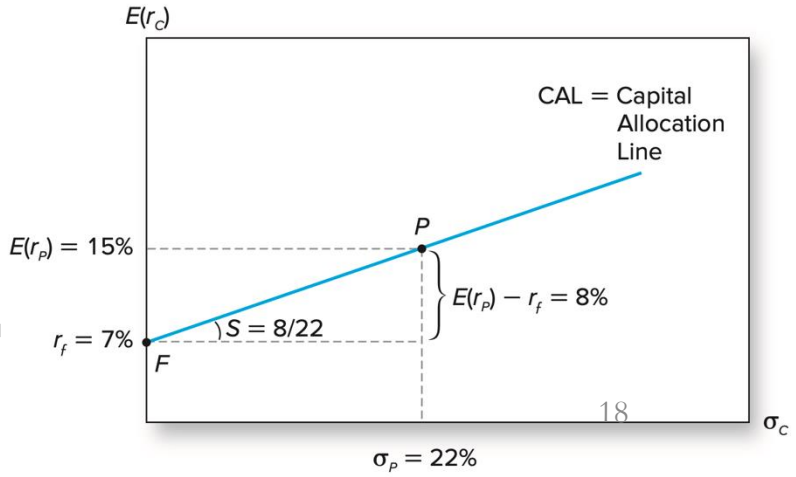
- We can find the weights w_D and w_E to maximize the Sharpe ratio. Then use this portfolio P and a risk-free asset to construct the CAL

$$\max_{w_D, w_E} S_P = \frac{E(r_P) - r_f}{\sigma_P}$$

$$\text{s.t.} \; r_P = w_D r_D + w_E r_E$$

$$\sigma_P = \left(w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2 w_D w_E Cov(r_D, r_E) \right)^{0.5}$$

$$w_D + w_E = 1$$



Optimal portfolio weights to maximize Sharpe ratio

- The optimal portfolio weights that maximize Sharpe ratio is

$$w_D^* = \frac{E(R_D)\sigma_E^2 - E(R_E)Cov(R_D,R_E)}{E(R_D)\sigma_E^2 + E(R_E)\sigma_D^2 - [E(R_D)+E(R_E)]Cov(R_D,R_E)}$$

$$w_E^* = 1 - w_D^*$$

  - $R_D = r_D - r_f$ and $R_E = r_E - r_f$ are excess returns

Derivation of optimal w_D^* to maximize Sharpe Ratio

- Write S_P^2 in terms of w_D only (using $w_D = 1 - w_E$)
- Take the derivative of S_P^2 with respect to w_D and set it to zero

$$\frac{dS_p^2}{dw_D} = 0$$

- Solve w_D from the first order condition and then we can show it takes the form in the previous slide

Question

- Question: What is the weight of D and E to maximize the Sharpe ratio? What is the corresponding Sharpe ratio? Suppose the risk-free rate is 5%.

- This risky portfolio is referred to as the *optimal risky portfolio*

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Correlation coefficient, ρ_{DE}		0.30	