DATASCI 385 Quantitative Finance

Lecture 7: Capital allocation to risky assets

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Suggested reading: Investments Ch 6

EMORY

Lecture plan

- Portfolios of one risky asset and a risk-free asset

- Risk diversification

Risk aversion and utility values

- **Risk-averse** investor penalizes the expected rate of return by the risk involved
  - A portfolio is more attractive when its expected return is higher and its risk is lower
- **Utility score** to compare competing portfolios based on expected return and risk of those portfolios

$$U = E(r) - \frac{1}{2}A\sigma^2$$

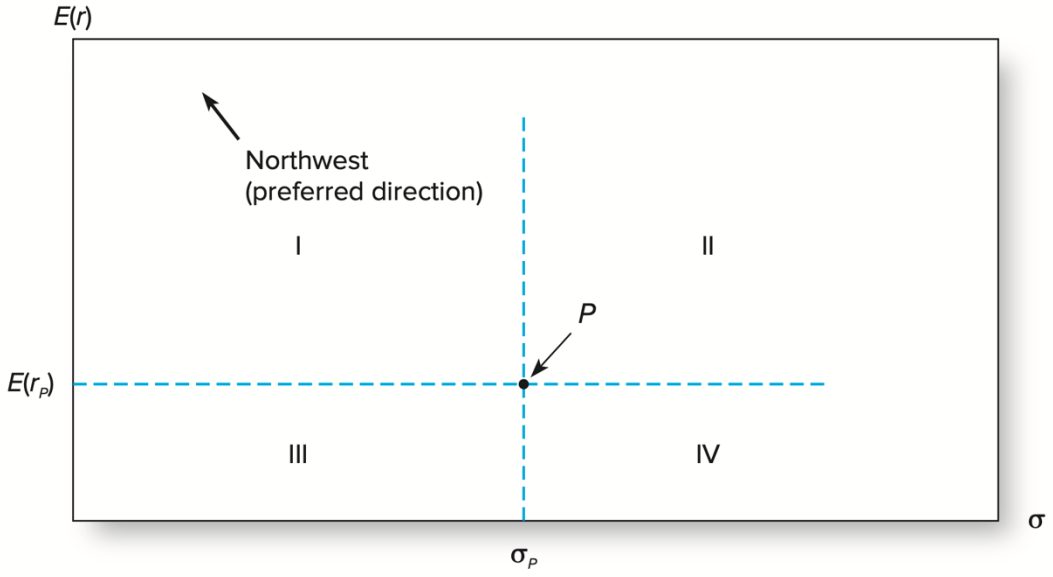
  - U: utility value
  - A: index of the investor's risk aversion
    - Risk-averse investors: $A > 0$
    - Risk-neural investors: $A = 0$
    - Risk-lover investors: $A < 0$
  - Quantify the rate at which investors are willing to trade off return against risk

Question

- Question: A portfolio has an expected rate of return of 20% and standard deviation of 30%. T-bills offer a safe rate of return of 7%. Would an investor with risk-aversion parameter $A = 4$ prefer to invest in T-bills or the risky portfolio? What if $A = 2$?

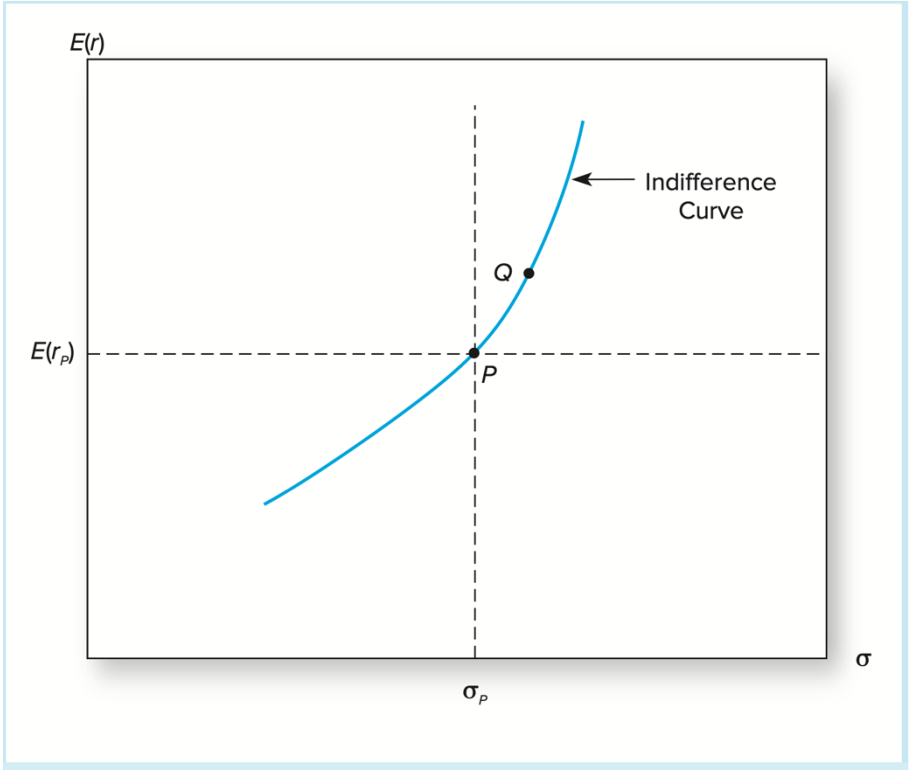
Risk and return tradeoff

- Portfolio *P* dominates any portfolio in quadrant IV
- Any portfolio in quadrant I dominates portfolio *P*
- Portfolios in quadrant II and III depend on investor's risk aversion



Indifference curve

- Indifference curve connects all portfolios—each representing a combination of expected return $E(r)$ and risk σ—that the same level of utility



Capital allocation across risky and risk-free portfolios

- **Asset allocation** of a complete portfolio C: y in the risky assets and $1 - y$ in the risk-free assets

- Example: total market value of a portfolio $\$300{,}000$
  - $\$90{,}000$ invested in risky assets
  - $\$210{,}000$ invested in risk-free assets
  - $y = \frac{210{,}000}{300{,}000} = .7$ (risky assets) and $1 - y = \frac{90{,}000}{300{,}000} = .3$ (risk-free assets)

- Treat all the risky assets as a portfolio, and all the risk-free assets a portfolio

Capital allocation across risky and risk-free portfolios

- **Asset allocation** of a complete portfolio C: y in the risky portfolio P and $1-y$ in the risk-free assets r_f
- For the risky portfolio P
  - Rate of return r_P
  - Expected return $E[r_P]$
  - Standard deviation σ_P
- For the complete portfolio C, expected return is

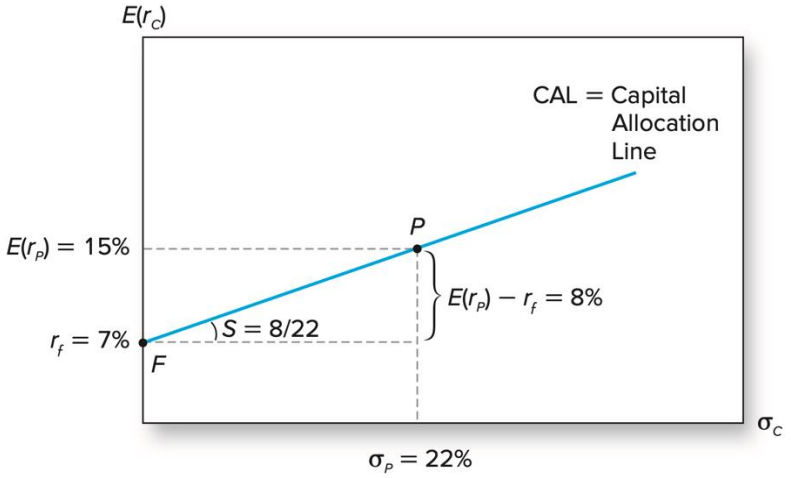
$$E[r_C] = E[yr_P + (1-y)r_f] = yE[r_P] + (1-y)r_f = r_f + y[E(r_P) - r_f]$$

and variance is

$$\sigma_C^2 = Var(yr_P + (1-y)r_f) = Var(yr_P) = y^2\sigma_P^2$$

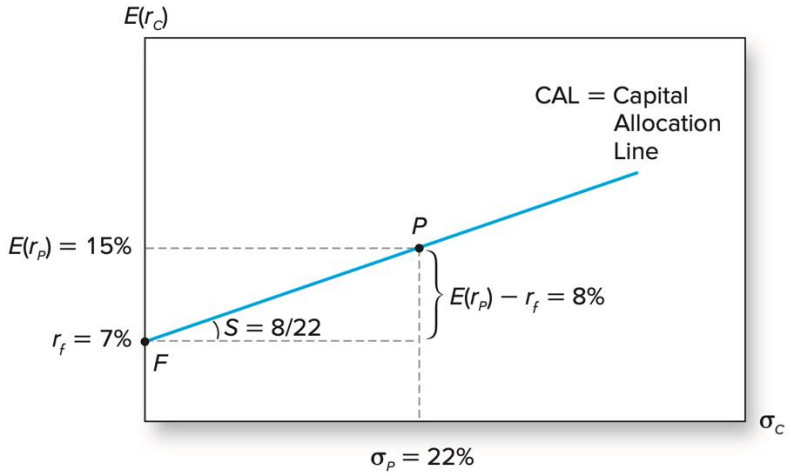
Capital allocation line (CAL)

- Capital allocation line (CAL): A graph showing all feasible risk–return combinations of a risky and risk-free asset
  - y-axis: $E[r_C] = r_f + y[E(r_P) - r_f]$
  - x-axis: $\sigma_C = y\sigma_P$
  - CAL: $E[r_C] = r_f + S \cdot \sigma_C$
    - To obtain this line, plug $y = \sigma_C/\sigma_P$ into $E[r_C] = r_f + y[E(r_P) - r_f]$
    - Then we obtain $E[r_C] = r_f + \frac{\sigma_C}{\sigma_P} \cdot [E(r_P) - r_f]$
    - Slope of the line (Sharpe ratio): $S = \frac{E(r_P) - r_f}{\sigma_P}$
  - Example: $E[r_P] = 15\%, \sigma_P = 22\%, r_f = 7\%$
    - $S = \frac{15-7}{22} = .36$
    - CAL: $E[r_C] = .07 + .36 \cdot \sigma_C$



Question

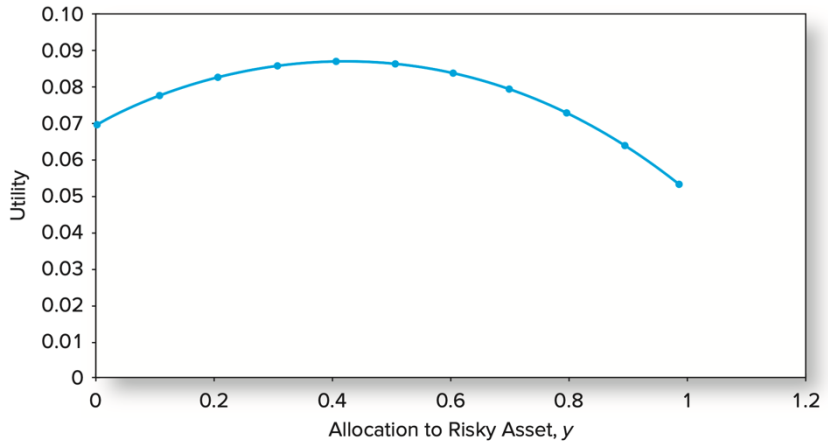
- Can the Sharpe ratio, $S = \frac{E(r_C) - r_f}{\sigma_C}$, of any combination of the risky asset and the risk-free asset be different from the ratio for the risky asset taken alone, $\frac{E(r_P) - r_f}{\sigma_P}$, which, in this case, is .36?



Risk tolerance and asset allocation

- Investors confronting CAL need to choose one optimal complete portfolio C (or one y) from the set of feasible choices
  - Investors maximize utility $U = E(r) - \frac{1}{2}A\sigma^2$
  - The expected return is $E[r_C] = r_f + y[E(r_P) - r_f]$
  - The variance is $\sigma_C^2 = y^2\sigma_P^2$
  - Plugging $E[r_C]$ and σ_C^2 into U, the decision is to choose y that maximizes

$$U = r_f + y[E(r_P) - r_f] - \frac{1}{2}Ay^2\sigma_P^2$$



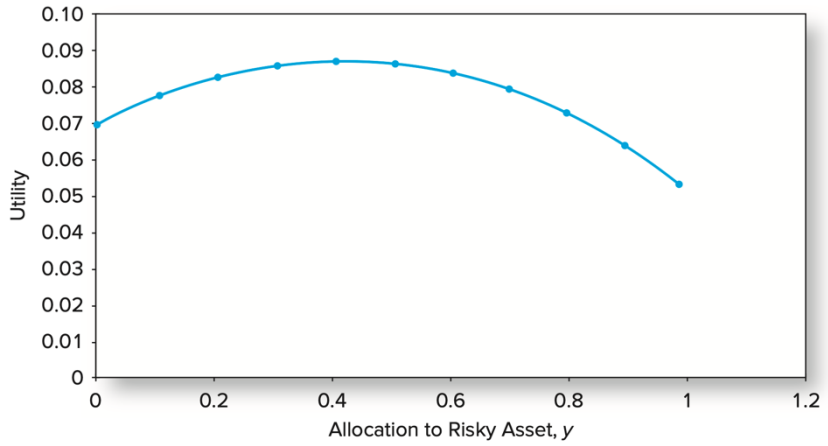
Solve the optimal portfolio

- Solve the y that maximizes

$$U = r_f + y\left[E(r_P) - r_f\right] - \frac{1}{2}Ay^2\sigma_P^2$$

- Take the derivative of U with respect to y and set the derivative to 0
  - $\left[E(r_P) - r_f\right] - Ay\sigma_P^2 = 0$

- Optimal y
  - $y^* = \frac{E(r_P) - r_f}{A\sigma_P^2}$

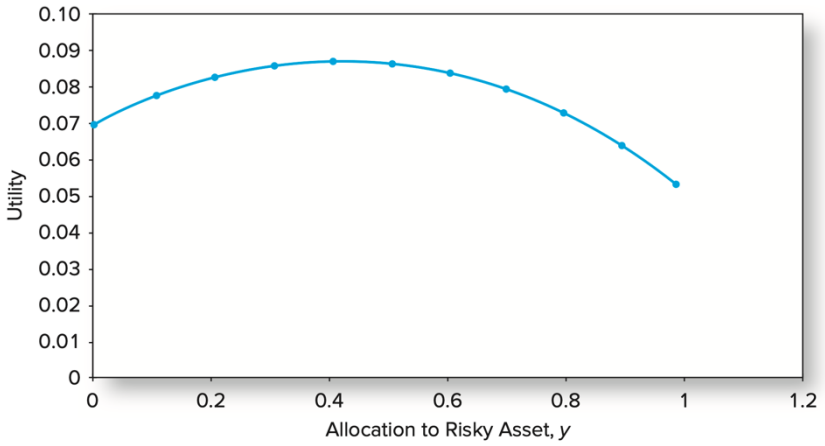


Example

- Suppose $r_f = 7\%, E(r_P) = 15\%, \sigma_P = 22\%$
- For an investor with risk aversion $A = 4$

$$y^* = \frac{E(r_P) - r_f}{A\sigma_P^2} = \frac{.15 - .07}{4 \times .22^2} = .41$$

  - 41% in the risky asset and 59% in the risk-free asset
  - $E[r_C] = 7 + [.41 \times (15 - 7)] = 10.28\%$
  - $\sigma_C = .41 \times 22 = 9.02\%$
  - $U = E[r_C] - \frac{1}{2} A\sigma_C^2 = 10.28\% - \frac{1}{2} \cdot 4 \cdot 9.02\%^2 = 0.08653$
  - Sharpe ratio is $\frac{E[r_C] - r_f}{\sigma_C} = \frac{3.28}{9.02} = .36$



Indifference curve analysis

- Indifference curve: all combinations of expected return $E(r)$ and volatility σ such that utility level is the same

$$U = E[r] - \frac{1}{2} \times A \times \sigma^2$$

- For example, an investor with risk aversion $A = 4$
  - Portfolio 1: Risk-free portfolio with $r_f = 5\%$
    - $U_1 = 0.05 - \frac{1}{2} \times 4 \times 0 = 0.05$
  - Portfolio 2: Risky portfolio with return $E[r] = .0502$ and $\sigma = 1\%$
    - $U_2 = 0.0502 - \frac{1}{2} \times 4 \times 1\% = 0.05$

➢Two portfolios are on the same indifference curve with utility $U = .05$

Indifference curve analysis

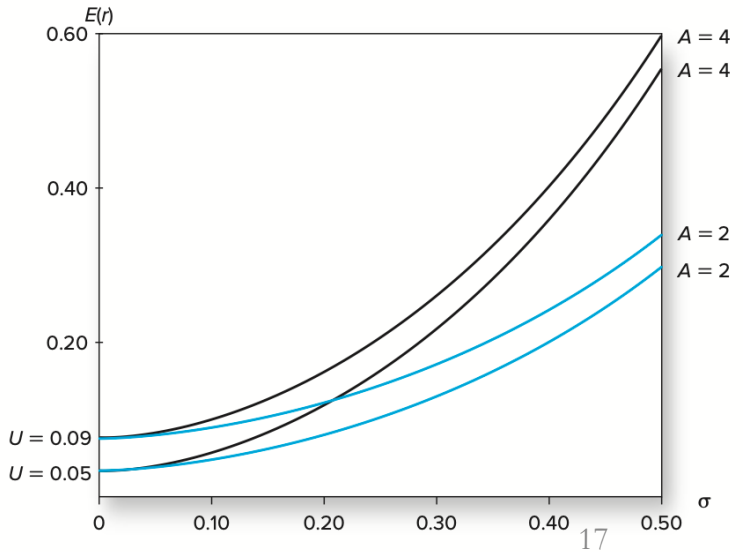
- Indifference curve: all combinations of expected return $E(r)$ and volatility σ such that utility level is the same

$$U = E[r] - \frac{1}{2} \times A \times \sigma^2$$

- Given U, identify the required $E[r]$ for every σ through

$$E[r] = U + \frac{1}{2} \times A \times \sigma^2$$

- Given the same σ, the required $E[r]$ for the less risk-averse investor (i.e., smaller A) is lower to achieve the same U

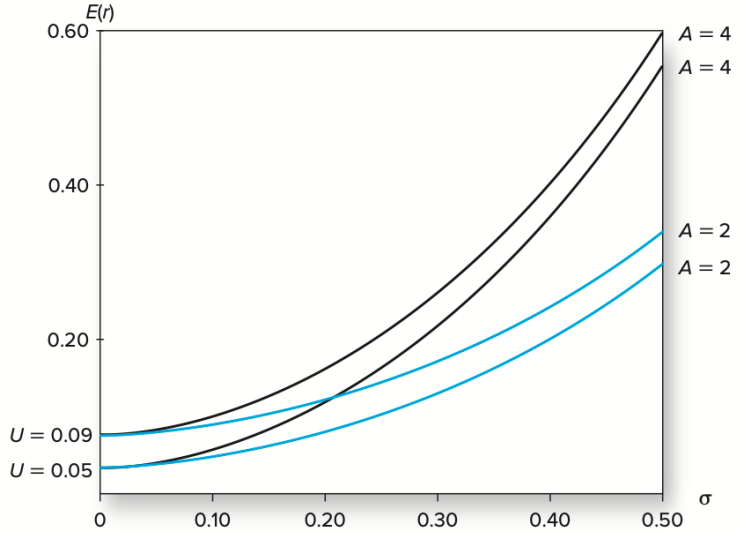


Comparative statics

- More risk-averse investors (i.e., larger A) have steeper indifference curves than less risk-averse investors
  - Steepness is measured by

$$\frac{dE[r]}{d\sigma} = \frac{d\left(U + \frac{1}{2} \times A \times \sigma^2\right)}{d\sigma} = A\sigma$$

  - Steepness increases with both A and σ, but does not vary with U
  - Steeper curves mean that per unit increase in σ requires larger increase in $E[r]$
- Higher indifference curves correspond to higher levels of utility



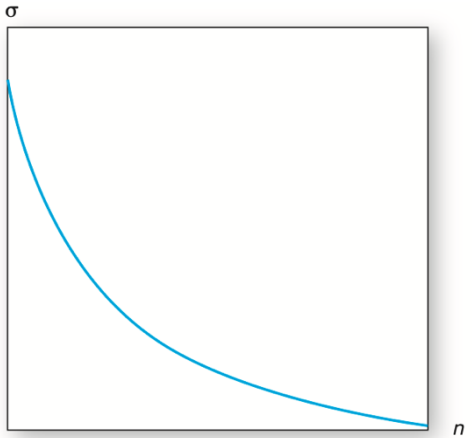
Diversification and portfolio risk

- Suppose your portfolio only contains one stock: Digital Computer Corp.
- Two sources of uncertainty:
  - Systematic risk: Risks from the condition of the general economy, e.g., business cycle, inflation, interest rates, exchange rates
  - Firm-specific risk: e.g., research and development and personnel changes

- A naïve **diversification** strategy: half of your funds in ExxonMobil and half in Digital
  - When oil prices fall, hurting ExxonMobil, computer prices might rise, helping Digital; Vice versa
  - Diversification reduces portfolio risk: Two effects offset and stabilize portfolio return

Diversification when all risk is firm-specific

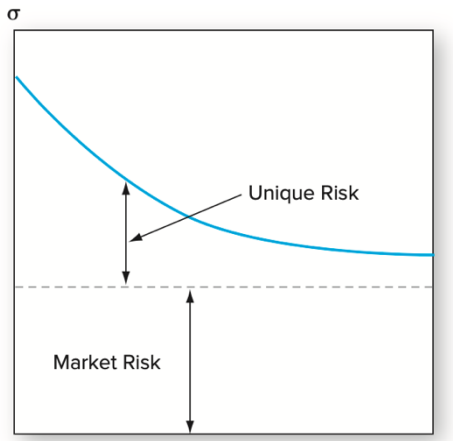
- If we diversify into many more securities, we spread out the exposure to firm-specific factors, and portfolio volatility falls
- When all risk is firm-specific, diversification can reduce risk to arbitrarily low levels. Firm-specific risk is also called diversifiable risk
- In portfolio P, weight of asset i is w_i. n assets in total
  - Expected return $E[r_P] = \sum_{i=1}^{n} w_i E[r_i]$
  - Variance $\sigma_P^2 = Cov\left(\sum_{i=1}^{n} w_i r_i, \sum_{j=1}^{n} w_j r_j\right) = \sum_{i,j=1}^{n} w_i w_j Cov(r_i, r_j)$
  - All risk is firm specific: $Cov(r_i, r_j) = 0$ for $i \neq j$
  - Suppose $Cov(r_i, r_i) = \sigma^2$ and $w_i = 1/n$
  - Then $\sigma_P^2 = n \cdot \frac{1}{n^2} \cdot \sigma^2 = \frac{1}{n} \cdot \sigma^2 \to 0$



Diversification when some risk is systematic

- If all stocks are affected by the business cycle (or other macroeconomic factors), we cannot avoid exposure to business cycle risk no matter how many stocks we hold
- Systematic risk can not be eliminated even with extensive diversification, also called as market risk or nondiversifiable risk
- In portfolio P, weight of asset i is w_i. n assets in total
  - Variance $\sigma_P^2 = Cov\left(\sum_{i=1}^{n} w_i r_i, \sum_{j=1}^{n} w_j r_j\right) = \sum_{i,j=1}^{n} w_i w_j Cov(r_i, r_j)$
  - Some risk is systematic: Suppose $Cov(r_i, r_j) = \rho\sigma^2$ for $i \neq j$
  - Suppose $Cov(r_i, r_i) = \sigma^2$ and $w_i = 1/n$
  - Then $\sigma_P^2 = n(n-1) \cdot \frac{1}{n^2} \cdot \rho\sigma^2 + n \cdot \frac{1}{n^2} \cdot \sigma^2$

$= \frac{n-1}{n} \rho\sigma^2 + \frac{1}{n} \cdot \sigma^2 \to \rho\sigma^2$



Power of diversification

- In our example, $\sigma_P^2 = \frac{n-1}{n}\rho\sigma^2 + \frac{1}{n}\cdot\sigma^2 \to \rho\sigma^2$
- $\rho = 0$: all risk is firm-specific, diversification is powerful
- $\rho = 0.4$: some risk is systematic, diversification is less powerful
- $\rho = 1$: all risk is systematic, diversification is useless

		ρ = 0		ρ = 0.40	
Universe Size *n*	Portfolio Weights *w* = 1/*n* (%)	Standard Deviation (%)	Reduction in σ	Standard Deviation (%)	Reduction in σ
1	100	50.00	14.64	50.00	8.17
2	50	35.36		41.83	
5	20	22.36	1.95	36.06	0.70
6	16.67	20.41		35.36	
10	10	15.81	0.73	33.91	0.20
11	9.09	15.08		33.71	
20	5	11.18	0.27	32.79	0.06
21	4.76	10.91		32.73	
100	1	5.00	0.02	31.86	0.00
101	0.99	4.98		31.86	