# DATASCI 385 Quantitative Finance

# Lecture 11: Arbitrage pricing theory

Instructor: Ruoxuan Xiong

Suggested reading: Investments Ch 10

EMORY

# Capital asset pricing model (CAPM)

- CAPM provides expected return-beta relationship in equilibrium for any asset $i$

$$E(R_i) = E(r_i) - r_f = \beta_i[E(r_M) - r_f]$$

  - An asset's appropriate risk premium $E(R_i)$ is proportional to its contribution to the risk of the overall portfolio, i.e.,

$$\frac{w_i E(R_i)}{w_i Cov(R_i,R_M)} = \frac{E(R_M)}{Cov(R_M,R_M)}, \text{ so } \beta_i = \frac{Cov(R_i,R_M)}{\sigma_M^2} \qquad \sigma_M^2 = Cov(R_M,R_M)$$

  - Firm-specific risk is not priced by the market

- Expected return-beta relationship for portfolio $P$ with weight $w_i$ in asset $i$ (with $\sum_i w_i = 1$ and $\sum_i w_i \beta_i = \beta_P$)

$$E(r_P) = w_1 E(r_1) + w_2 E(r_2) + \cdots + w_n E(r_n) = \sum_{i=1}^{n} w_i\left(r_f + \beta_i[E(r_M) - r_f]\right) = r_f + \beta_P\,[E(r_M) - r_f]$$

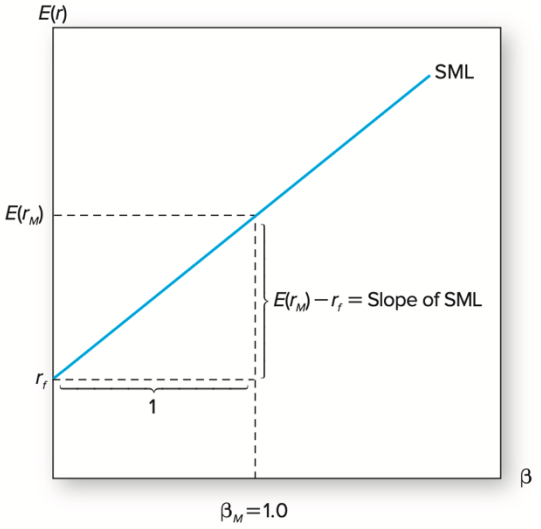
# Security market line

- Security's risk premium is proportional to both beta and risk premium of the market portfolio, i.e.,

$$\beta[E(r_M) - r_f]$$

- Security market line (SML): expected return-beta relationship $E(r)$ vs $\beta$
  - Slope is $E(r_M) - r_f$
  - Risk is measured by $\beta$
  - Graphs *individual asset* risk premiums as a function of asset risk

(held as parts of well-diversified portfolios, i.e., contribution to portfolio variance)



# Single-index model

- The single index model describes how excess returns $R_i$ move, which is the sum of three parts

$$R_i = \alpha_i + \beta_i R_M + e_i$$

  - Market-wide factors: $\beta_i R_M$
  - Nonmarket premium: $\alpha_i$
  - Firm-specific outcomes: $e_i$
- The expectation has $E(R_i) = \alpha_i + \beta_i E(R_M)$

- Based on CAPM, the equilibrium value of $\alpha_i$ is 0. Otherwise, if $\alpha_i > 0$, investors will buy the stocks, bid up the prices, and expected return will be lower

# Example of alpha and beta

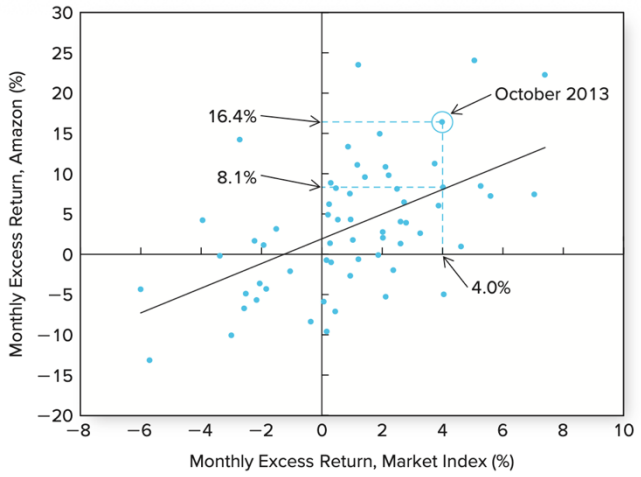



**Figure 8.1** Scatter diagram for Amazon against the market index and Amazon's security characteristic line, 5 years ending June 2018

**Table 8.1**

Excel output: Regression statistics for Amazon's SCL

| Regression Statistics | |
|---|---|
| Multiple *R* | 0.5351 |
| *R*-Square | 0.2863 |
| Adjusted *R*-Square | 0.2742 |
| Standard Error | 0.0686 |
| Observations | 60 |

| | Coefficients | Standard Error | *t*-statistic | *p*-value |
|---|---|---|---|---|
| Intercept | 0.0192 | 0.0093 | 2.0645 | 0.0434 |
| Market index | 1.5326 | 0.3150 | 4.8648 | 0.0000 |

| Ticker | Company | Beta | Alpha | *R*-Square | Residual Std Dev | Standard Error Beta | Standard Error Alpha | Adjusted Beta |
|---|---|---|---|---|---|---|---|---|
| CPB | Campbell Soup | 0.247 | 0.001 | 0.012 | 0.064 | 0.292 | 0.009 | 0.498 |
| NEM | Newmont Mining | 0.430 | 0.005 | 0.011 | 0.117 | 0.538 | 0.016 | 0.620 |
| MCD | McDonald's | 0.563 | 0.006 | 0.165 | 0.036 | 0.165 | 0.005 | 0.709 |
| SBUX | Starbucks | 0.581 | 0.004 | 0.112 | 0.046 | 0.212 | 0.006 | 0.721 |
| KO | Coca-Cola | 0.663 | −0.001 | 0.258 | 0.032 | 0.146 | 0.004 | 0.776 |
| UNP | Union Pacific | 0.859 | 0.005 | 0.212 | 0.047 | 0.216 | 0.006 | 0.906 |
| PFE | Pfizer | 0.895 | −0.000 | 0.367 | 0.033 | 0.153 | 0.005 | 0.930 |
| XOM | ExxonMobil | 0.920 | −0.006 | 0.342 | 0.036 | 0.166 | 0.005 | 0.946 |
| MSFT | Microsoft | 0.923 | 0.012 | 0.193 | 0.054 | 0.246 | 0.007 | 0.948 |
| INTC | Intel | 0.934 | 0.007 | 0.187 | 0.055 | 0.254 | 0.007 | 0.956 |
| GOOG | Alphabet (Google) | 0.960 | 0.008 | 0.209 | 0.053 | 0.243 | 0.007 | 0.973 |
| DIS | Walt Disney | 1.288 | −0.001 | 0.496 | 0.037 | 0.169 | 0.005 | 1.192 |
| BAC | Bank of America | 1.357 | 0.003 | 0.270 | 0.063 | 0.291 | 0.009 | 1.238 |
| BA | Boeing | 1.368 | 0.011 | 0.330 | 0.055 | 0.254 | 0.007 | 1.246 |
| AMZN | Amazon | 1.533 | 0.019 | 0.286 | 0.069 | 0.315 | 0.009 | 1.355 |
| MRO | Marathon Oil | 2.632 | −0.023 | 0.314 | 0.110 | 0.506 | 0.015 | 2.088 |
| | AVERAGE | 1.010 | 0.003 | 0.235 | 0.057 | 0.260 | 0.008 | 1.006 |
| | STD DEVIATION | 0.560 | 0.009 | 0.127 | 0.025 | 0.115 | 0.003 | 0.374 |

**Table 8.2**

Index model estimates: Regressions of excess stock returns on the excess return of the broad market index over 60 months, five years ending June 2018

# Adjusted beta

- Beta coefficients seem to move toward 1 over time
  - A new firm may be more unconventional
  - As a firm grows, it often diversifies and resembles the rest of the economy more

$$\text{Adjusted } \beta = \frac{2}{3} \times \text{estimated } \beta + \frac{1}{3} \times 1$$

- Shrink toward 1, the average $\beta$

| Ticker | Company | Beta | Alpha | *R*-Square | Residual Std Dev | Standard Error Beta | Standard Error Alpha | Adjusted Beta |
|---|---|---|---|---|---|---|---|---|
| CPB | Campbell Soup | 0.247 | 0.001 | 0.012 | 0.064 | 0.292 | 0.009 | 0.498 |
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# Predicting beta

- Use data on beta in different periods and estimate a regression equation

$$\text{Forecast } \beta = a + b_1 \times \text{current } \beta + b_2 \times \text{firm size} + b_3 \times \text{debt ratio} + \cdots$$

  - Rosenberg and Guy found (1) variance of earnings, (2) variance of cash flow, (3) growth in earnings per share, (4) market capitalization (firm size), (5) dividend yield, (6) debt-to-asset ratio, helpful to predict beta

**Table 8.3**

Industry betas and adjustment factors

| Industry | Beta | Adjustment Factor |
|---|---|---|
| Agriculture | 0.99 | −0.140 |
| Drugs and medicine | 1.14 | −0.099 |
| Energy utilities | 0.60 | −0.237 |
| Gold | 0.36 | −0.827 |
| Construction | 1.27 | 0.062 |
| Air transport | 1.80 | 0.348 |
| Trucking | 1.31 | 0.098 |
| Consumer durables | 1.44 | 0.132 |

# Factor models of security returns

- Uncertainty in asset returns has two sources
  - A common or macroeconomic factor
  - Firm-specific events

- Single factor model

$$R_i = E(R_i) + \beta_i F + e_i$$

  - $E(R_i)$: the expected excess return on stock $i$
  - $F$: deviation of the common factor from its expected value. Therefore, $E(F) = 0$. It measures new information concerning the macroeconomy
  - $e_i$: nonsystematic components of returns (related to firm-specific events, uncorrelated with $F$)

# Example of the single factor model

- Single factor model: $R_i = E(R_i) + \beta_i F + e_i$
  - For example, $F$ is macro factor (news about the state of the business cycle, measure by the unexpected percentage change in gross domestic product ($\Delta GDP$)
  - The consensus is that GDP will increase by $4\%$ this year
  - Suppose also that a stock's $\beta_i = 1.2$
  - If GDP increases by only $3\%$, then $F = \Delta GDP = 3\% - 4\% = -1\%$
  - The stock is $1.2\%$ lower ($\beta_i F = 1.2 \times (-1\%) = -1.2\%$) than previously expected

# Factor models

- Confining systematic risk to a single factor is restrictive
  - Extra-market sources of risk may arise from interest rates, inflation, and other sources of uncertainty
- Multifactor models can better describe security returns
- For example, a two-factor model

$$R_i = E(R_i) + \beta_{i,GDP} \cdot \Delta GDP + \beta_{i,IR} \cdot \Delta IR + e_i$$

  - $\Delta IR$: Unexpected change in interest rates

# The two-factor model

- $R_i = E(R_i) + \beta_{i,GDP} \cdot \Delta GDP + \beta_{i,IR} \cdot \Delta IR + e_i$

- Regulated electric-power utility firm in a mostly residential area
  - $\beta_{i,GDP}$ is low
  - $\beta_{i,IR}$ is high

- An airline firm
  - $\beta_{i,GDP}$ is high
  - $\beta_{i,IR}$ is low

# Example

- Suppose we estimate the two-factor model for Northeast Airlines and find
  - $R_i = .133 + 1.2 \cdot \Delta GDP - .3 \cdot \Delta IR + e_i$
  - Expected excess return $E(R_i)$ is 13.3%
  - If GDP increases 1% beyond current expectation ($\Delta GDP = 1\%$), return on Northeast's shares increases on average by 1.2%
  - If increase rates increase 1% beyond current expectation ($\Delta IR = 1\%$), return on Northeast's shares decreases on average by .3%

# Multi-factor model

- A description of the factors that affect security returns

$$R_i = E(R_i) + \beta_{i,GDP} \cdot \Delta GDP + \beta_{i,IR} \cdot \Delta IR + e_i$$

- There is no "theory" in the equation
- Where $E(R_i)$ comes from? What determines a security's expected excess rate of return?
- Arbitrage pricing theory helps to determine $E(R_i)$ in equilibrium

# Arbitrage pricing theory

- Stephen Ross developed the arbitrage pricing theory (APT) in 1976

- Like the CAPM, the APT predicts a security market line linking expected returns to risk

- Three key propositions
  1. Security returns can be described by a factor model
  2. There are sufficient securities to diversify away idiosyncratic risk
  3. Well-functioning security markets do not allow arbitrage opportunities to persist

# Law of one price

- If two assets are equivalent in all economically relevant respects, then they should have the same market price

- If they observe a violation of the law, they will engage in arbitrage activity—simultaneously buying the asset where it is cheap and selling where it is expensive
  - E.g., different prices of a stock on two different exchanges
  - Involve long–short positions

# Diversification in a single-factor security market

- If a portfolio is well diversified, its firm-specific or nonfactor risk becomes negligible
- A portfolio with $n$ stocks, each with weight $w_i$ and $\sum_i w_i = 1$

$$R_P = E(R_P) + \beta_P F + e_P$$

  - $\beta_P = \sum w_i \beta_i$
  - $E(R_P) = \sum w_i E(R_i)$
  - Nonsystematic return $e_P = \sum w_i e_i$
- Portfolio variance can be decomposed into two parts

$$\sigma_P^2 = Var(\beta_P F + e_P) = \beta_P^2 Var(F) + Var(e_P) \qquad Cov(F, e_P) = 0$$

  - If the portfolio is equally weighted $w_i = 1/n$, then

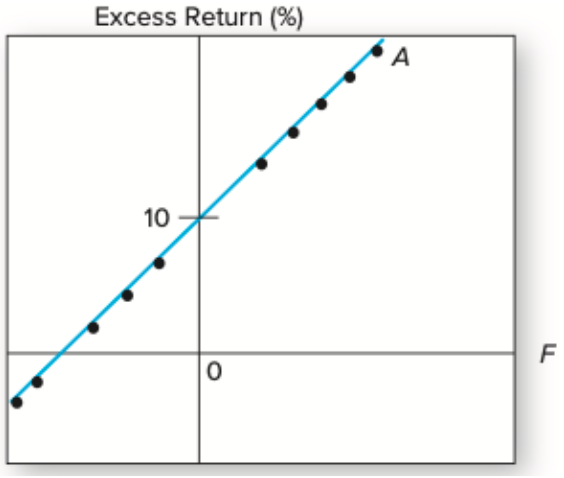
$$Var(e_P) = Var(\sum w_i e_i) = \sum w_i^2 Var(e_i) = \frac{1}{n} \sum \frac{Var(e_i)}{n} = \frac{1}{n} \overline{Var(e_i)} \to 0 \Rightarrow e_P \to 0$$

# Well-diversified portfolios

- For a well-diversified portfolio

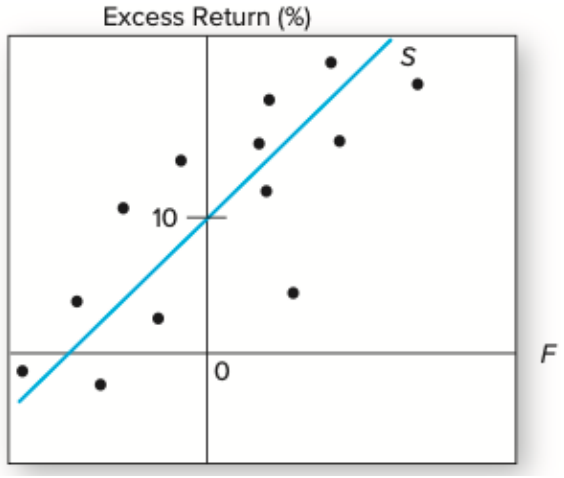
$$R_P = E(R_P) + \beta_P F$$

Portfolio return




$R_A = 10\% + 1.0 \times F$

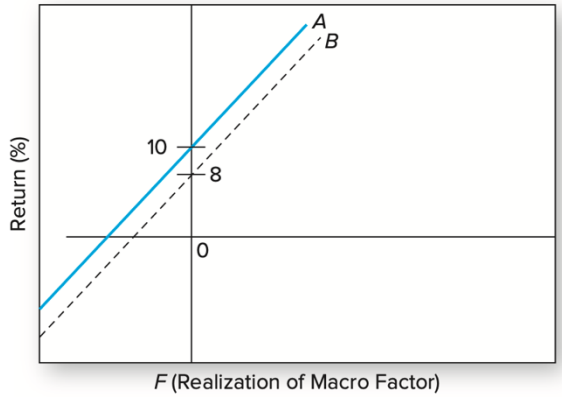
Stock return




$R_S = 10\% + 1.0 \times F + e_S$

# The security market line of the APT

- Only the systematic or factor risk of a portfolio of securities should be related to its expected returns
- All well-diversified portfolios with the same beta must have the same expected return
- If $\beta_A = \beta_B$, then $E(R_A) = E(R_B)$. Otherwise, an arbitrage opportunity exists
  - sell short \$1 million of B
  - buy \$1 million of A
  - a zero-net-investment strategy




$E(R_A) = 10, E(R_B) = 8$

# The security market line of the APT

- Their risk premiums must be proportional to beta
  - Otherwise, an arbitrage opportunity exists

- If the excess return on a well-diversified portfolio $R$ follows

$$R_P = E(R_P) + \beta_P F$$

- Then the risk premium is

$$E(R_P) = \beta_P E(F)$$

# Single-factor APT vs CAPM

- The APT serves many of the same functions as the CAPM
- APT gives us a benchmark for rates of return
- APT highlights the crucial distinction between nondiversifiable risk (factor risk), which requires a reward in the form of a risk premium, and diversifiable risk, which does not
- APT does not require that almost all investors be mean-variance optimizers. Rely on a highly plausible assumption that precludes arbitrage opportunities

# Single-factor APT and CAPM

- The CAPM provides a statement on the expected return–beta relationship for all securities

- APT implies that this relationship holds for all but perhaps a small number of securities, as APT is built on well-diversified portfolios

# Multifactor APT and risk premium

- Consider a two-factor model for portfolio $P$

$$R_P = E(R_P) + \beta_P F = E(R_P) + \beta_{P1} F_1 + \beta_{P2} F_2$$

  - $F = [F_1, F_2]'$ and $\beta_P = [\beta_{P1}, \beta_{P2}]$

  - For example
    - $F_1 = \Delta GDP$: the departure of GDP growth from expectations
    - $F_2 = \Delta IR$: the unanticipated change in interest rates

  - Factor portfolios: benchmark (well-diversified) portfolios (or tracking portfolios) in the APT, have a beta of one on one of the factors and a beta of zero on any other factor

# Risk premium for multifactor APT

- $E(R_1) = E(r_1) - r_f$: risk premium of the first factor portfolio
- $E(R_2) = E(r_2) - r_f$: risk premium of the second factor portfolio

- The risk premium of any portfolio $P$

$$E(R_P) = \beta_P E(F) = \beta_{P1} \cdot E(R_1) + \beta_{P2} \cdot E(R_2)$$
$$\Rightarrow E(r_P) = r_f + \beta_{P1} \cdot \left(E(r_1) - r_f\right) + \beta_{P2} \cdot \left(E(r_2) - r_f\right)$$

# Example

- Suppose that the two factor portfolios have expected returns $E(r_1) = 10\%$ and $E(r_2) = 12\%$ respectively and $r_f = 4\%$

- A well-diversified portfolio $P$ has $\beta_{P1} = .5$ and $\beta_{P2} = .75$. The expected return of $P$ equals

$$E(r_P) = r_f + \beta_{P1} \cdot \left(E(r_1) - r_f\right) + \beta_{P2} \cdot \left(E(r_2) - r_f\right)$$
$$= 4 + .5 \times (10 - 4) + .75 \times (12 - 4) = 13\%$$