DATASCI 385 Quantitative Finance

Lecture 10: Capital asset pricing model

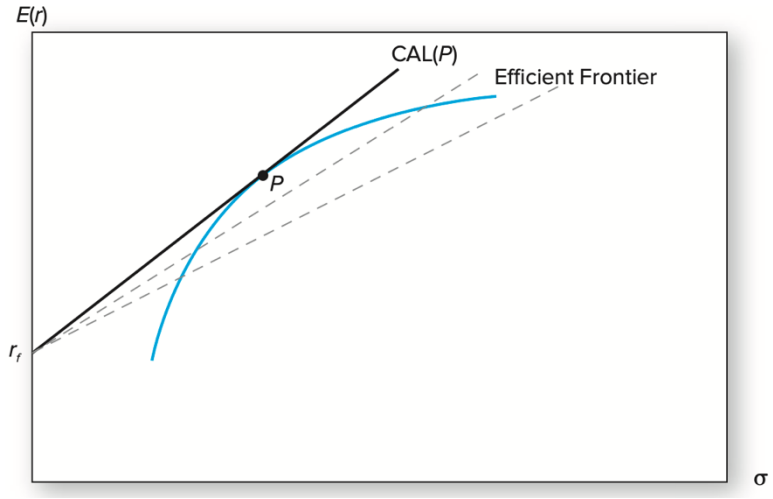
Instructor: Ruoxuan Xiong

Suggested reading: Investments Ch 9

EMORY

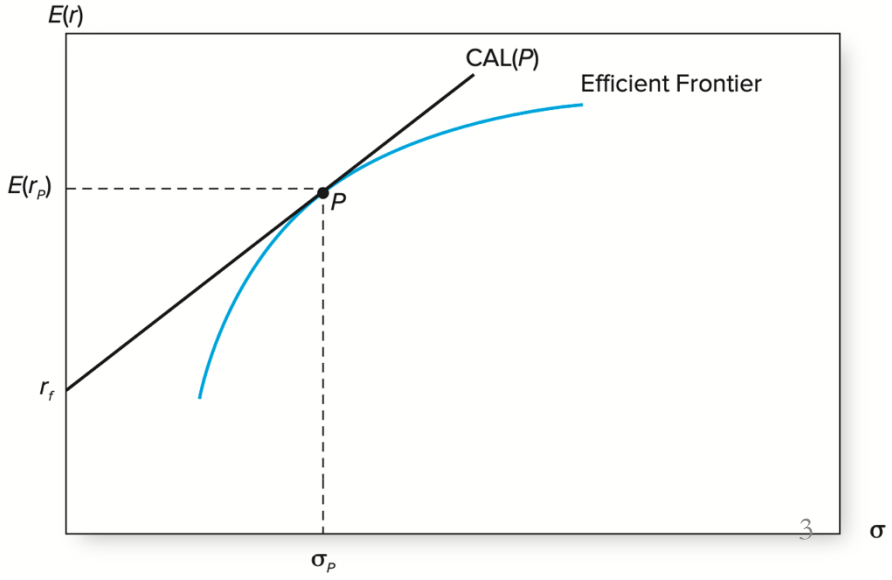
Implication of Markowitz model with risk-free assets

- Identify the point with the highest Sharpe ratio on the efficient frontier. This portfolio is denoted as P (shared across all investors)
- Construct the CAL that connects the risk-free asset with P
- The CAL is tangent to the efficient frontier
- For individual investors, choose the optimal complete portfolio with y in P and $1 - y$ in the risk-free asset to maximize $U = E[r_C] - \frac{1}{2} A\sigma_C^2$



Market portfolio

- Suppose all investors choose the same P with the same set of weights for each risky asset
- This portfolio is the market portfolio, denoted as M, i.e., the value-weighted portfolio of all assets in the investable universe
- Two assumptions for all investors to choose the same P



Assumption 1 is on individual investors

- Investors are rational, mean-variance optimizers
- Their common planning horizon is a single period
- Investors all use identical input lists (e.g., mean and covariance), an assumption often termed homogeneous expectations
  - Homogeneous expectations are consistent with the assumption that all relevant information is publicly available

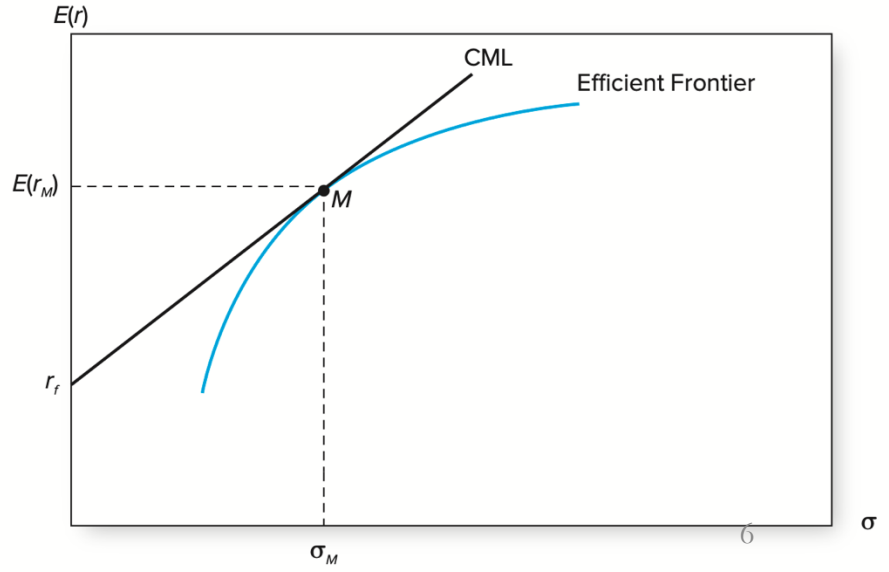
Assumption 2 is on market structure

- All assets are publicly held and trade on public exchanges
- Investors can borrow or lend at a common risk-free rate, and they can take short positions on traded securities
- No taxes
- No transaction costs

Capital market line

- Capital market line (CML): The capital allocation line (CAL) based on the market portfolio *M*

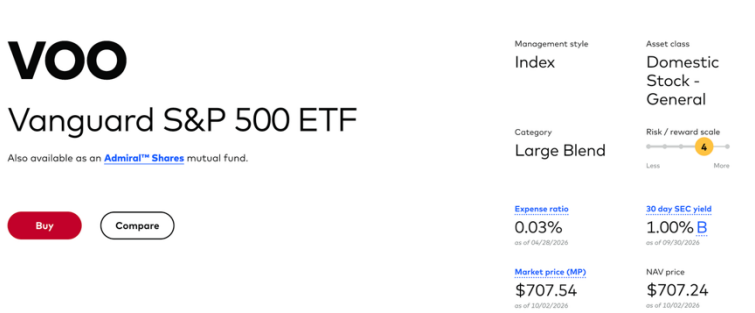
- Price adjustment process: If investors do not include the stock of some company in the portfolio, the price of this stock drops and then becomes attractive to be included in the portfolio

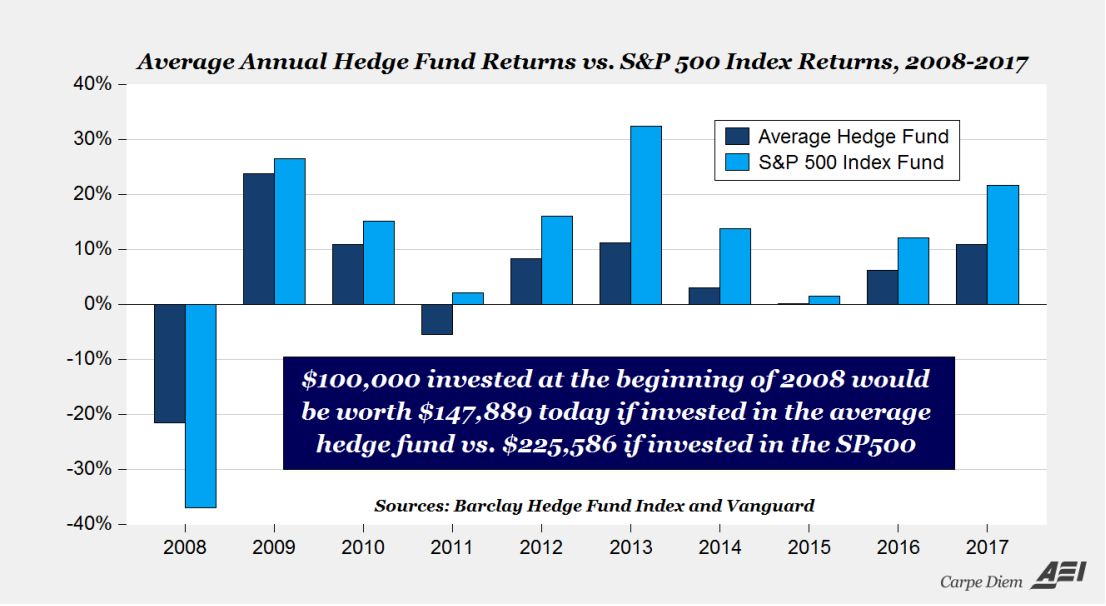


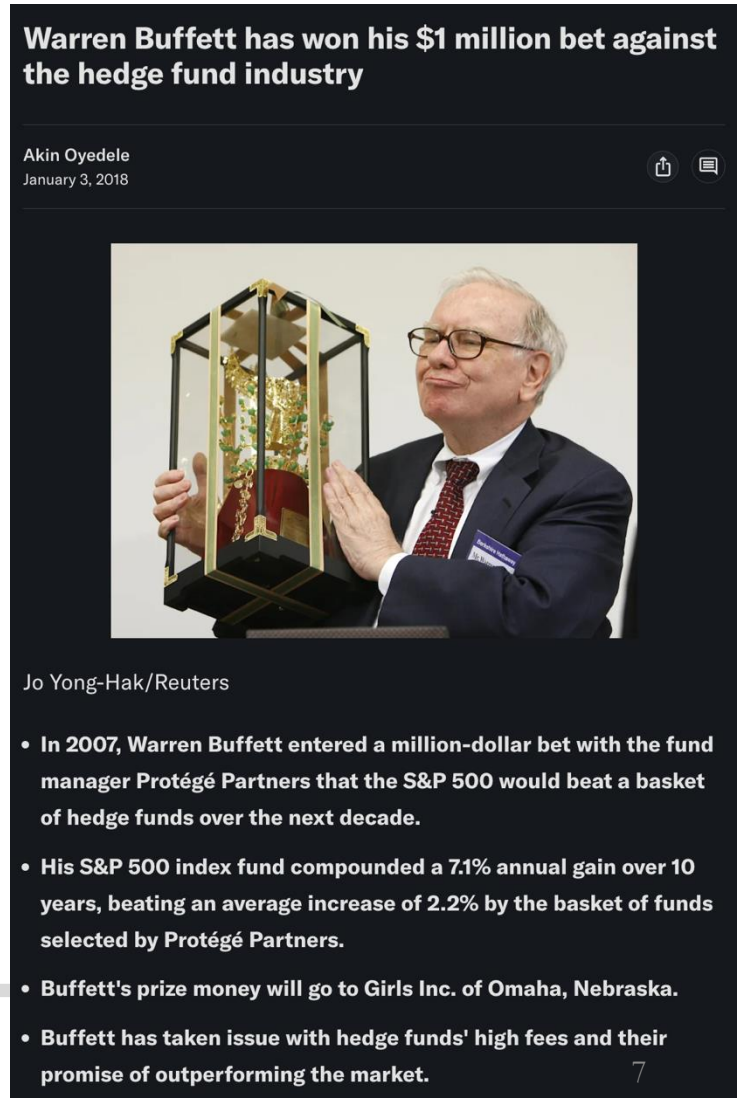
Implication of market portfolio

- **Mutual fund theorem**: investing in a market-index portfolio is efficient (passive strategy)
  - Broad diversification
  - Expense ratio is low

vs "2 and 20 model" for hedge fund







https://www.berkshirehathaway.com/letters/2016ltr.pdf

Risk premium of the market portfolio

- Individual investor with risk aversion A chooses a proportion y, allocated to the optimal portfolio M, to maximize utility

$$max_y U = E(r_C) - \frac{1}{2} A\sigma_C^2$$
$$= r_f + y\left[E(r_M) - r_f\right] - \frac{1}{2} Ay^2\sigma_M^2$$

  - Optimal $y^* = \frac{E(r_M) - r_f}{A\sigma_M^2} = \frac{E[R_M]}{A\sigma_M^2}$, where $R_M = r_M - r_f$

- Net borrowing and lending across all investors is zero, and the average position in M is 100%, or $\bar{y} = 1$
- Let a market representative investor be the one with $\bar{y} = 1$ and risk aversion $\bar{A}$. Then the risk premium of the market portfolio is

$$\frac{E[R_M]}{A\sigma_M^2} = \bar{y} = 1 \Rightarrow E(R_M) = \bar{A}\sigma_M^2$$

Some facts

- Data from the last nine decades for the broad U.S. equity market
  - Risk premium $E(R_M)$: 8.3%
  - Standard deviation σ_M: 20.1%

- Using $E(R_M) = \bar{A}\sigma_M^2$, the average risk aversion $\bar{A}$ is
  - $\bar{A} = \frac{E(R_M)}{\sigma_M^2} = \frac{8.3\%}{20.1\%^2} = 2.0544$

- Sharpe ratio
  - $S = \frac{E(R_M)}{\sigma_M} = 0.413$

Capital asset pricing model

- Capital asset pricing model (CAPM): a prediction of the relationship between the risk of an asset and its expected return
- The risk premium on an asset is determined by its contribution to the risk of investors' overall portfolios (i.e., market portfolio)
- Example: GE's stock
  - GE's contribution to the risk premium of the market portfolio is $w_{GE}E(R_{GE})$
  - GE's contribution to the variance of the market portfolio is

$$w_{GE}Cov(R_M, R_{GE}) = w_{GE}Cov\left(\sum_{i=1}^{n} w_i R_i, R_{GE}\right) = w_{GE}\sum_{i=1}^{n} w_i Cov(R_i, R_{GE})$$

Portfolio Weights	w_1	w_2	...	w_{GE}	...	w_n
w_1	$Cov(R_1, R_1)$	$Cov(R_1, R_2)$	...	$Cov(R_1, R_{GE})$	...	$Cov(R_1, R_n)$
w_2	$Cov(R_2, R_1)$	$Cov(R_2, R_2)$	...	$Cov(R_2, R_{GE})$	...	$Cov(R_2, R_n)$
⋮	⋮	⋮		⋮		⋮
w_{GE}	$Cov(R_{GE}, R_1)$	$Cov(R_{GE}, R_2)$	...	$Cov(R_{GE}, R_{GE})$	...	$Cov(R_{GE}, R_n)$
⋮	⋮	⋮		⋮		⋮
w_n	$Cov(R_n, R_1)$	$Cov(R_n, R_2)$	...	$Cov(R_n, R_{GE})$	...	$Cov(R_n, R_n)$

Risk-to-reward ratios of GE and market portfolio

- The reward-to-risk ratio for investments in GE is

$$\frac{\text{GE's contribution to risk premium}}{\text{GE's contribution to variance}} = \frac{w_{GE}E(R_{GE})}{w_{GE}Cov(R_M, R_{GE})} = \frac{E(R_{GE})}{Cov(R_M, R_{GE})}$$

- The reward-to-risk ratio for market portfolio is (market price of risk)

$$\frac{\text{Market risk premium}}{\text{Market variance}} = \frac{E(R_M)}{\sigma_M^2}$$

- A basic principle of equilibrium: All investments should offer the same reward-to-risk ratio

$$\frac{E(R_{GE})}{Cov(R_M, R_{GE})} = \frac{E(R_M)}{\sigma_M^2}$$

Risk premium of a stock

- The risk premium of GE is

$$E(R_{GE}) = \frac{Cov(R_M, R_{GE})}{\sigma_M^2} E(R_M)$$

$$\Rightarrow E(r_{GE}) = r_f + \beta_{GE}\left[E(r_M) - r_f\right] \text{ (expected return–beta relationship)}$$

  - $\beta_{GE} = Cov(R_M, R_{GE})/\sigma_M^2$: contribution of GE to the variance of market portfolio as a fraction of the total variance of the market portfolio

- Expected return–beta relationship for any asset i

$$E(r_i) = r_f + \beta_i\left[E(r_M) - r_f\right]$$

Expected return-beta relationship

- Expected return-beta relationship for any asset i

$$E(r_i) = r_f + \beta_i[E(r_M) - r_f]$$

  - Sum of risk-free rate r_f (compensation for "waiting") and a risk premium $\beta_i[E(r_M) - r_f]$ (compensation for "worrying")
  - Firm-specific risk is not priced by the market

- Expected return-beta relationship for portfolio P with weight w_i in asset i

$$E(r_P) = w_1E(r_1) + w_2E(r_2) + \cdots + w_nE(r_n)$$
$$= \sum_i^n w_i(r_f + \beta_i[E(r_M) - r_f]) = r_f + \beta_P[E(r_M) - r_f]$$

Market portfolio

- For the market portfolio

$$E(r_M) = r_f + \beta_M[E(r_M) - r_f]$$

  - This implies that $\beta_M = 1$
  - The weighted average value of beta across all assets is 1

Question

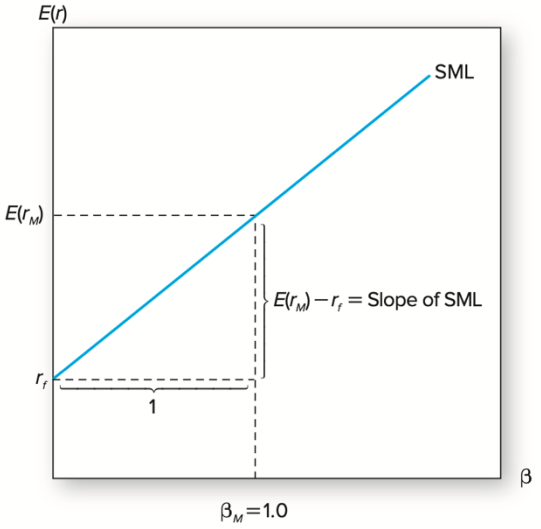
- Suppose that the risk premium on the market portfolio is estimated at 8% with a standard deviation of 22%. What is the risk premium on a portfolio invested 25% in Toyota and 75% in Ford if they have betas of 1.10 and 1.25, respectively?

Security market line

- Security's risk premium is proportional to both beta and risk premium of the market portfolio, i.e.,

$$\beta[E(r_M) - r_f]$$

- Security market line (SML): expected return-beta relationship $E(r)$ vs β
  - Slope is $E(r_M) - r_f$
  - Graphs *individual asset* risk premiums as a function of asset risk (held as parts of well-diversified portfolios, i.e., contribution to portfolio variance)



Interpretation of β in security market line

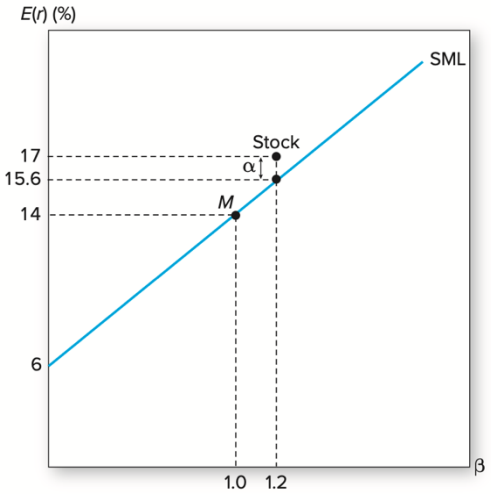
- In the security market line,

$$\beta[E(r_M) - r_f]$$

  - If $\beta = 0$, then $E(r_i) = r_f$ and the risk is diversifiable
  - If $\beta < 0$, then $E(r_i) < r_f$ and payoffs in bad times (insurance)
  - If $\beta > 0$, then $E(r_i) > r_f$ and the risk is compensated

Mispricing relative to security market line

- If assets are "fairly priced", then they are exactly on the SML. All securities must lie on the SML in market equilibrium
- If a stock is underpriced, then it will plot above the SML
- The difference between the fair and actually expected rate of return on a stock is called the stock's alpha, denoted by α
- For example, $\alpha = 1.4\%$ for the stock in the figure



Question

- Stock XYZ has an expected return of 12% and risk of $\beta = 1$. Stock ABC has expected return of 13% and $\beta = 1.5$. The market's expected return is 11%, and $r_f = 5\%$

- According to the CAPM, which stock is a better buy?