State-Varying Factor Models of Large Dimensions

Markus Pelger and Ruoxuan Xiong

Stanford University

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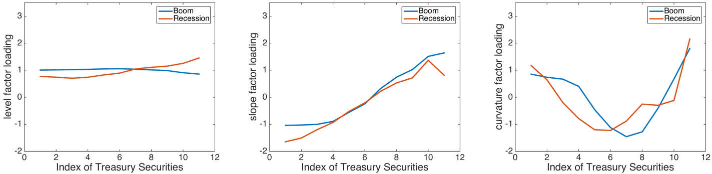
Motivation

- Conventional large-dimensional latent factor model assumes the exposures to factors (factor loadings) are constant over time
- Observation: Asset prices' exposures to the market (and other risk factors) are time-varying
- Example: Term-structure factor exposure is different in recessions and booms.

Figure: PCA Factor Loadings for Treasuries in Boom and Recession



(a) Level Factor (b) Slope Factor (c) Curvature Factor

This paper

Research Question:

1. Find latent factors and loadings that are state-dependent.
2. Test if factor model is state-dependent.

Key elements of estimator

1. Statistical factors instead of pre-specified (and potentially miss-specified) factors
2. Uses information from large panel data sets: Many cross-section units with many time observations
3. Factor structure can be time-varying as a general non-linear function of the state process

Contribution of this paper

Contribution

- Theoretical
  - PCA estimator combined with kernel projection for factors, state-varying factor loadings and common components
  - Inferential theory for estimators for $N, T \to \infty$:
    - consistency
    - asymptotic normal distribution and standard errors
  - Test for state-dependency of latent factor model
    - Generalized correlation test statistic detects for which states model changes
    - Non-standard superconsistency
- Empirical
  - State-dependency of factor loadings in US Treasury securities

Literature (partial list)

- Large-dimensional factor models with constant loadings
  - Bai (2003): Distribution theory
  - Fan et al. (2013): Sparse matrices in factor modeling
- Large-dimensional factor models with time-varying loadings
  - Su and Wang (2017): Local time-window
  - Pelger (2018), Aït-Sahalia and Xiu (2017): High-frequency
  - Fan et al. (2016): Projected PCA
- Large-dimensional factor models with structural breaks
  - Stock and Watson (2009): Inconsistency
  - Breitung and Eickmeier (2011), Chen et al. (2014): Detection

The Model

State-varying factor model

- X_{it} is the observed data for the i-th cross-section unit at time t
- State variable S_t at time t

$$X_{it} = \underbrace{\underset{1\times r}{\Lambda_i(S_t)}}_{\text{loadings}} \underbrace{\underset{r\times 1}{F_t}}_{\text{factors}} + \underbrace{e_{it}}_{\text{idiosyncratic}} \qquad i = 1, \cdots N,\ t = 1, \cdots T$$

  - N cross-section units (large), time horizon T (large)
  - r systematic factors (fixed)
- Factors F, loadings $\Lambda(S_t)$, idiosyncratic components e are unknown
- Data X and state process S_t observed

The Model

Examples (with one factor) equivalent to multi-factor representation

- Loadings linear in state: $\Lambda_i(S_t) = \Lambda_{i,1} + \Lambda_{i,2} S_t$

$$X_{it} = \Lambda_{i,1} \underbrace{F_t}_{F_{t,1}} + \Lambda_{i,2} \underbrace{(S_t F_t)}_{F_{t,2}} + e_{it}$$

- Loadings nonlinear in discrete state: $\Lambda_i(S_t) = g_i(S_t)$, $S_t \in \{s_1, s_2\}$

$$X_{it} = \underbrace{g_i(s_1)}_{\Lambda_{i,1}} \underbrace{\mathbb{1}_{\{S_t=s_1\}} F_t}_{F_{t,1}} + \underbrace{g_i(s_2)}_{\Lambda_{i,2}} \underbrace{\mathbb{1}_{\{S_t=s_2\}} F_t}_{F_{t,2}} + e_{it}$$

Our model

- Loadings nonlinear in non-discrete state: $\Lambda_i(S_t) = g_i(S_t)$ with continuous distribution function for S_t

$\Rightarrow$ Cumbersome/No multi-factor representation

The Model: Main Assumptions

Approximate state-varying factor model

- Systematic factors explain a large portion of the variance
- Idiosyncratic risk is nonsystematic: Weak time-series and cross-sectional correlation
- State: recurrent (infinite observations around the state to condition on) with continuous stationary PDF
- Factor Loadings: deterministic functions of the state and the functions are Lipschitz continuous (observations in the nearby state are useful)
  $\exists C, \ \|\Lambda_i(s + \Delta s) - \Lambda_i(s)\| \leq C|\Delta s|$

The Model: Extension

Robustness to noise in state process

- State process is observed with noise:

$$\underbrace{X_t}_{N\times 1} = \underbrace{\Lambda(S_t)}_{N\times r} \underbrace{F_t}_{r\times 1} + \underbrace{\mathcal{E}_t}_{N\times r} \underbrace{F_t}_{r\times 1} + \underbrace{e_t}_{N\times 1} = \Lambda(S_t)F_t + \psi_t + e_t$$

- Under weak conditions noise can be treated like idiosyncratic noise.

$\Rightarrow$ All results hold!

Missing relevant states

- Assume loadings depend on multiple states but we only condition on a subset of them.
- State-varying factor model explains strictly more variance than constant loading model.

$\Rightarrow$ More parsimonious representation even under misspecification.

The Model: Intuition

Intuition for Estimation

- Constant loadings:
  Loadings are principal components of covariance matrix

$$Cov(X_t) = \Lambda Cov(F_t)\Lambda^\top + Cov(e_t).$$

- State-varying loadings:
  Loadings for $S_t = s$ are principal components of covariance matrix conditioned on the state $S_t = s$:

$$Cov(X_t|S_t = s) = \Lambda(s)Cov(F_t|S_t = s)\Lambda(s)^\top + Cov(e_t|S_t = s).$$

$\Rightarrow$ Intuition: Estimate conditional covariance matrix $Cov(X_t|S_t = s)$ with kernel projection and apply PCA to it.

The Model: Nonparametric Estimation

Obtective function and nonparametric estimation

The estimators minimize mean squared error conditioned on state:

$$\hat{F}^s, \hat{\Lambda}(s) = \underset{F^s, \Lambda(s)}{\arg\min} \frac{1}{NT(s)} \sum_{i=1}^{N} \sum_{t=1}^{T} K_s(S_t)(X_{it} - \Lambda_i(s)' F_t)^2$$

- Kernel function $K_s(S_t) = \frac{1}{h} K\left(\frac{S_t - s}{h}\right)$ (e.g. $K(u) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{u^2}{2}\}$)
- $T(s) = \sum_{t=1}^{T} K_s(S_t)$, $\frac{T(s)}{T} \xrightarrow{p} \pi(s)$ (stationary density of $S_t = s$)
- Bandwidth parameter h determines local “state window”

The Model: Nonparametric Estimation

Nonparametric estimation

- Project square root of kernel on the data and factors

$$X_{it}^s = K_s^{1/2}(S_t)X_{it} \qquad F_t^s = K_s^{1/2}(S_t)F_t$$

- PCA solves optimization problem

$$\hat{F}^s, \hat{\Lambda}(s) = \arg\min_{F^s, \Lambda(s)} \frac{1}{NT(s)} \sum_{i=1}^{N} \sum_{t=1}^{T} (X_{it}^s - \Lambda_i(s)' F_t^s)^2$$

$\Rightarrow$ Apply PCA to conditional covariance matrix

- $\hat{F}^s$ are the eigenvectors corresponding to top k eigenvalues of estimated conditional covariance matrix $\frac{1}{NT(s)}(X^s)'X^s$
- $\hat{\Lambda}(s)$ are coefficients from regressing X^s on $\hat{F}^s$

The Model: Nonparametric Estimation

Major challenge: Bias term

$$X_t^s = \Lambda(S_t)F_t^s + e_t^s = \underbrace{\Lambda(s)F_t^s + e_t^s}_{\bar{X}_t^s} + \underbrace{(\Lambda(S_t) - \Lambda(s))F_t^s}_{\Delta X_t^s}.$$

- $\Delta X_{it}^s = \Lambda_i(S_t)F_t^s - \Lambda_i(s)F_t^s = O_p(h)$
- Kernel bias complicates problem and lowers convergence rates

Theorem: Consistency

Assume $N, Th \to \infty$ and $\delta_{NT,h}h \to 0$ with $\delta_{NT,h} = min(\sqrt{N}, \sqrt{Th})$:

$$\delta_{NT,h}^2 \left(\frac{1}{T} \sum_{t=1}^T \left\| \hat{F}_t^s - (H^s)^T F_t^s \right\|^2 \right) = O_p(1)$$

$$\delta_{NT,h}^2 \left(\frac{1}{N} \sum_{i=1}^N \left\| \hat{\Lambda}_i(s) - (H^s)^{-1} \Lambda_i(s) \right\|^2 \right) = O_p(1)$$

for known full rank matrix H^s

Limiting Distribution of Estimated Factors

Theorem (Factors)

Assume $\sqrt{Nh}/(Th) \to 0$, $Nh \to \infty$ and $Nh^2 \to 0$. Then

$$\begin{aligned} & \sqrt{N}\left(K_s^{-1/2}(S_t)\hat{F}_t^s - (H^s)'F_t\right) \\ = \quad & (V_r^s)^{-1}\frac{(\hat{F}^s)'F^s}{T}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\Lambda_i(s)e_{it} + o_p(1) \\ \xrightarrow{D} \quad & N(0,(V^s)^{-1}Q^s\Gamma_t^s(Q^s)'(V^s)^{-1}) \end{aligned}$$

- Rotation matrix $H^s = \frac{\Lambda(s)'\Lambda(s)}{N}\frac{(F^s)'\hat{F}^s}{T}(V_r^s)^{-1}$
- $K_s^{-1/2}(S_t)\hat{F}_t^s$ converges to some rotation of F_t at rate $\sqrt{N}$
- Efficiency mainly depends on asymptotic distribution of $\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\Lambda_i(s)e_{it}$

Limiting Distribution of Estimated Factor Loadings

Theorem (Loadings)

Assume $\sqrt{Th}/N \to 0$, $Th \to \infty$, and $Th^3 \to 0$. Then

$$\begin{aligned} & \sqrt{Th}(\hat{\Lambda}_i(s) - (H^s)^{-1}\Lambda_i(s)) \\ = \quad & (V_r^s)^{-1} \frac{(\hat{F}^s)' F^s}{Th} \frac{\Lambda(s)'\Lambda(s)}{N} \frac{\sqrt{Th}}{T(s)} \sum_{t=1}^{T} F_t^s e_{it}^s + o_p(1) \\ \xrightarrow{D} \quad & N(0, ((Q^s)')^{-1} \Phi_i^s (Q^s)^{-1}) \end{aligned}$$

- $\hat{\Lambda}_i(s)$ converges to some rotation of $\Lambda_i(s)$ at rate $\sqrt{Th}$
- Efficiency mainly depends on asymptotic distribution of $\frac{\sqrt{Th}}{T(s)} \sum_{t=1}^{T} F_t^s e_{it}^s = \frac{\sqrt{Th}}{T(s)} \sum_{t=1}^{T} K_s(S_t) F_t e_{it}$

Limiting Distribution of Common Component

Theorem (Common Components)

Assume $Nh \to \infty$, $Th \to \infty$, $Nh^2 \to 0$ and $Th^3 \to 0$. Then for each i

$$\delta_{NT,h}(\hat{C}_{it,s} - C_{it,s}) = \frac{\delta_{NT,h}}{\sqrt{N}} \Lambda_i(s)' \Sigma_{\Lambda(s)}^{-1} \left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \Lambda_i(s) e_{it} \right) + \frac{\delta_{NT,h}}{\sqrt{Th}} F_t' \Sigma_{F|s}^{-1} \left(\frac{\sqrt{Th}}{T(s)} \sum_{t=1}^{T} F_t^s e_{it}^s \right) + o_p(1)$$

- $\delta_{NT,h} = min(\sqrt{N}, \sqrt{Th})$
- Define $C_{it,s} = F_t' \Lambda_i(s)$ and $\hat{C}_{it,s} = (\frac{\hat{F}_t^s}{K_s^{1/2}(S_t)})' \hat{\Lambda}_i(s)$
- If $N/(Th) \to 0$, $\Lambda_i(s) e_{it}$ dominates
- If $Th/N \to 0$, $F^s(t) e_{it}^s$ dominates

Generalized Correlation

Test for constancy: Generalized correlation test

Consider loadings in two states $\Lambda_1 = \Lambda(s_1)$ and $\Lambda_2 = \Lambda(s_2)$. Test for

$$\mathcal{H}_0 : \Lambda_1 = \Lambda_2 G \text{ for some full rank square matrix } G$$
$$\mathcal{H}_1 : \Lambda_1 \neq \Lambda_2 G \text{ for any full rank square matrix } G$$

- Generalized correlation, defined as ρ invariant of G

$$\rho = trace\left\{ \left(\frac{\Lambda_1^T \Lambda_1}{N}\right)^{-1} \left(\frac{\Lambda_1^T \Lambda_2}{N}\right) \left(\frac{\Lambda_2^T \Lambda_2}{N}\right)^{-1} \left(\frac{\Lambda_2^T \Lambda_1}{N}\right) \right\}$$

- $\hat{\rho}$ estimated ρ and r is #factors
- Equivalent to test $\mathcal{H}_0 : \rho = r$ and $\mathcal{H}_1 : \rho < r$

Generalized Correlation

Theorem: Generalized correlation test

Assume $\sqrt{N}/(Th) \to 0$, $Nh \to \infty$, $Th \to \infty$, $\sqrt{Th}/N \to 0$, $Nh^2 \to 0$ and $NTh^3 \to 0$:

$$\sqrt{NTh}(\hat{\rho} - r - \hat{\xi}^\top \hat{b}) \xrightarrow{d} N(0, \Omega)$$

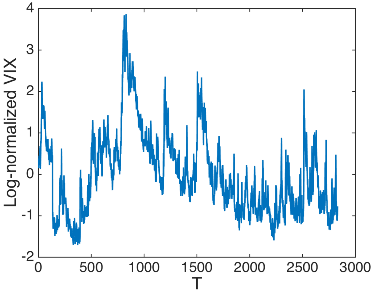
- $\xi^\top b$ bias term with feasible estimates $\hat{b}$ and $\hat{\xi}$
- feasible estimator for asymptotic covariance $\hat{\Omega}$

$\Rightarrow$ Superconsistent rate $\sqrt{NTh}$ (corner case)

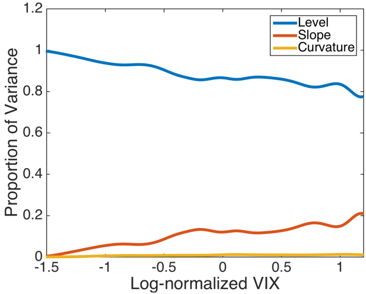
- $h \in [1/T^{1/2}, 1/T^{3/4}]$: combinations of N and T exist to satisfy the rate conditions

Empirical Applications

- US Treasury Securities Yields from 2001-07-31 to 2016-12-01: $N = 11$, $T = 2832$: 1, 3, 6 mo., 1, 2, 3, 5, 7, 10, 20, 30 yr.
- State: Log-normalized VIX
- Generalized correlation: $\hat{\rho}(\Lambda(Boom), \Lambda(Recession)) = 2.6352$
  $\Rightarrow$ reject $\rho \approx 3$ for $\Lambda(Boom) \approx \Lambda(Recession)$



(a) Log-normalized VIX

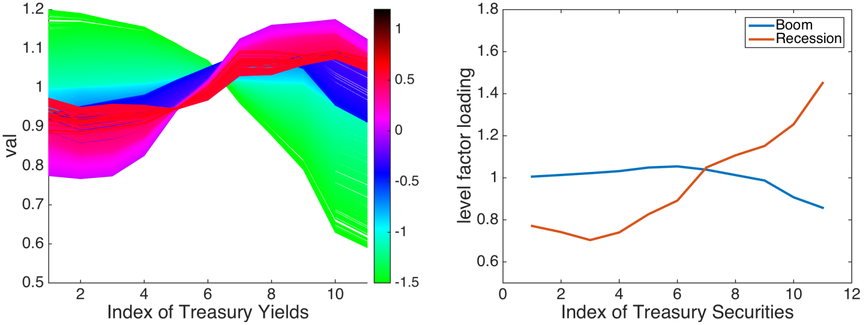


(b) Proportion of variance explained

Empirical Applications

- Long term bonds have higher weights in the level factor in high VIX/recession

Figure: Factor Loading to the Level Factor (1st Factor)

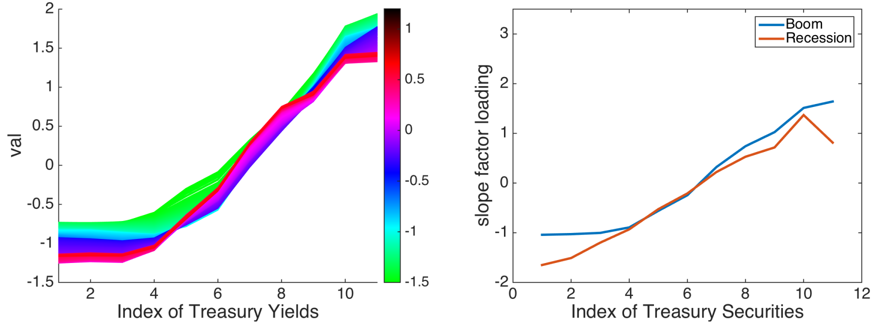


(a) Log-normalized VIX (b) Recession Indicator

Empirical Applications

- In high vix/recession: short term bonds more negative and long term bonds less positive

Figure: Factor Loading to the Slope Factor (2nd Factor)

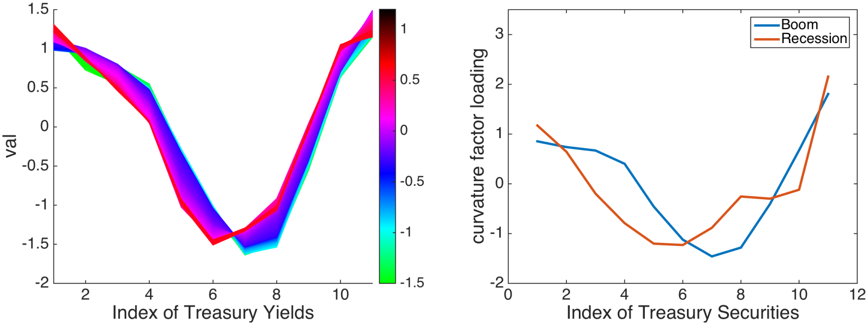


(a) Log-normalized VIX (b) Recession Indicator

Empirical Applications

- Minimum portfolio weight in the curvature factor shifts to shorter term bond in high vix/recession

Figure: Factor Loading to the Curvature Factor (3rd Factor)

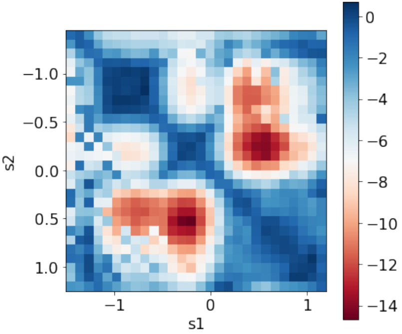


(a) Log-normalized VIX (b) Recession Indicator

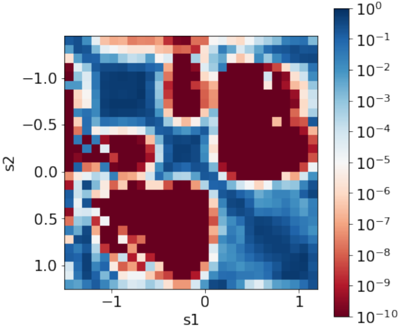
Empirical Applications: Test Constancy of Loadings

- Loadings in low vix are different from loadings in high vix (red region)

Figure: Generalized Correlation Test of Estimated Loadings in Two States under Null Hypothesis ($\mathcal{H}_0$: Loadings in Two States are Constant)



(a) t-value



(b) p-value

S&P500 Stock Returns

- Daily stocks returns (01/2004 to 12/2016): $N = 332$ and $T = 3253$
- State: Log-normalized VIX

$\Rightarrow$ Constant loading model needs roughly three more factors to explain the same variation in- and out-of-sample.

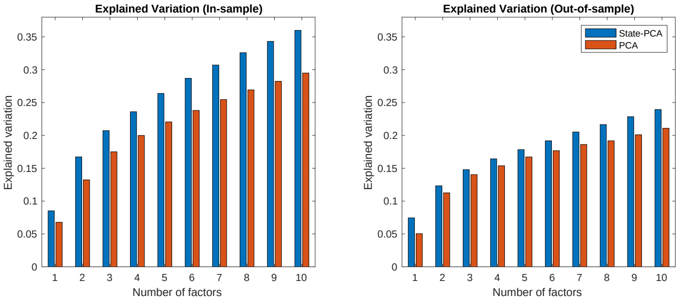


Figure: Variation explained by state-varying and constant loading model.

S&P500 Stock Returns

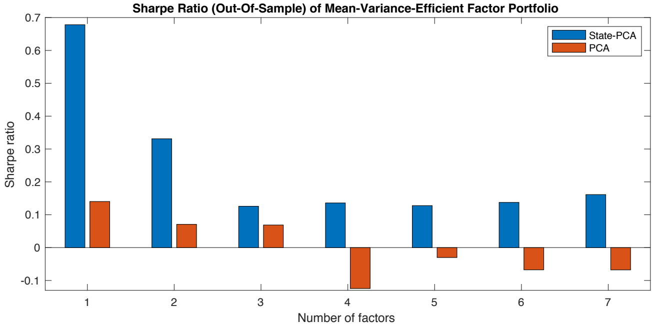


Figure: Out-of-sample Sharpe ratio of mean-variance efficient portfolio based on latent factors of the state-varying and constant loading model.

$\Rightarrow$ State-varying factor models capture more pricing information than constant-loading factors

Conclusion

Methodology

- Estimators for latent factors, loadings and common components where loadings are state-dependent
- We combine large dimensional factor modeling with nonparametric estimation
- Asymptotic properties of the estimators
- Constancy test for estimated state-varying factor loadings

Empirical Results

- We discover the movements of factor loadings by state values in the US Treasury Securities and Equity Markets
- Promising empirical results in other data sets

Data Generating Process for Simulations

- We generate data from a one-factor model

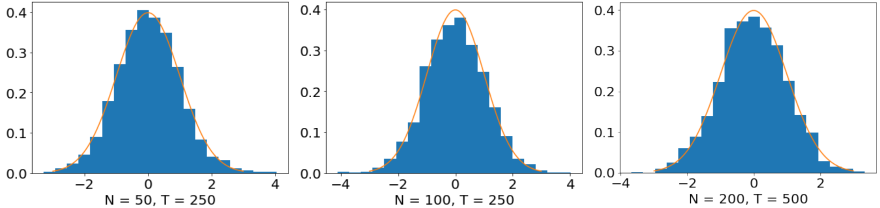
$$X_{it} = \Lambda_i(S_t)F_t + e_{it}$$

- Factor: $F_t \sim N(0, 1)$
- State: Ornstein–Uhlenbeck (OU) process (mean-reverting) $S_t = \theta(\mu - S_t)d_t + \sigma dW_t$, where $\theta = 1$, $\mu = 0.2$, and $\sigma = 1$
  - stochastic volatility in financial data
- Loading: $\Lambda_i(S_t) = \Lambda_{0i} + \frac{1}{2}S_t\Lambda_{1i} + \frac{1}{4}S_t^2\Lambda_{2i} + \frac{1}{8}S_t^3\Lambda_{3i}$, where $\Lambda_{0i}, \Lambda_{1i}, \Lambda_{2i}, \Lambda_{3i} \sim N(0, 1)$
- Idiosyncratic errors: IID/Heteroskedasticity/Cross sectional dependence

Simulation of CLT for Estimated Factors

- $\sqrt{N}(\hat{\Gamma}_t^s)^{-1/2}(\hat{Q}^s)^{-1}\hat{V}^s\left(K_s^{-1/2}(S_t)\hat{F}_t^s - (H^s)'F_t\right) \xrightarrow{d} N(0, I_r)$

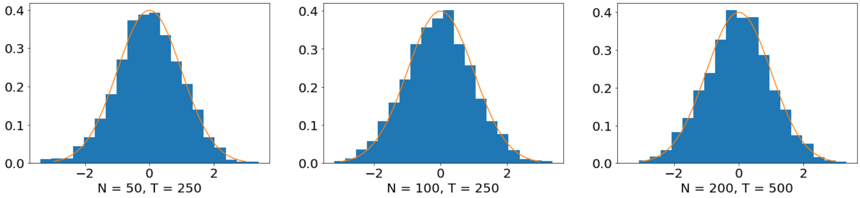
Figure: Comparison between simulated normalized factor distribution and standard normal distribution



Simulation of CLT for Estimated Loadings

- $\sqrt{Th}(\hat{\Phi}_i^s)^{-1/2}(\hat{Q}^s)'(\hat{\Lambda}_i(s) - (H^s)^{-1}\Lambda_i(s)) \xrightarrow{d} N(0, I_r)$

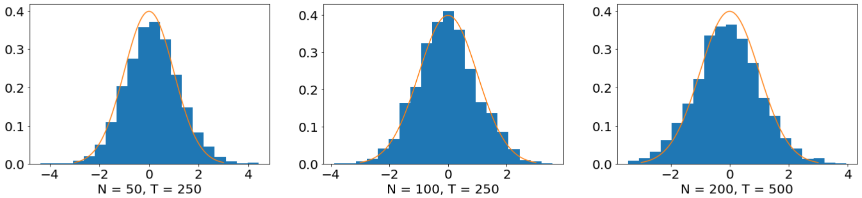
Figure: Comparison between simulated normalized loading distribution and standard normal distribution



Simulation of CLT for Common Component

- $\left(\frac{1}{N}\hat{V}_{it,s} + \frac{1}{Th}\hat{W}_{it,s}\right)^{-1/2}\left(\hat{C}_{it,s} - C_{it,s}\right) \xrightarrow{d} N(0, I_r)$

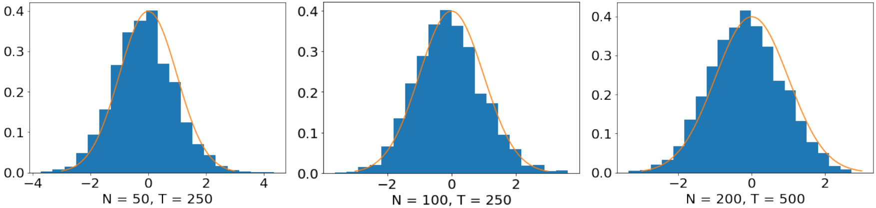
Figure: Comparison between simulated normalized common component distribution and standard normal distribution



Simulation of CLT for Estimated Generalized Correlation

- Loading: constant with the state $\Lambda_i(S_t) = \Lambda_{0i}$
- $\sqrt{NTh}(\hat{\rho} - r - \hat{\xi}^T \hat{b})/(\hat{\Omega})^{1/2} \xrightarrow{d} N(0,1)$

Figure: Comparison between simulated normalized estimated generalized correlation distribution and standard normal distribution



Recover Functional Form of Loadings vs. State

- $\Lambda_i(S_t) = \Lambda_{0i} + \frac{1}{2} S_t \Lambda_{1i} + \frac{1}{4} S_t^2 \Lambda_{2i} + \frac{1}{8} S_t^3 \Lambda_{3i}$

Figure: Loading as a function of the State