State-Varying Factor Models of Large Dimensions

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Motivation

- Conventional large-dimensional latent factor model assumes the exposures to factors (factor loadings) are constant over time
- Observation: Asset prices' exposures to the market (and other risk factors) are time-varying
- Example: Term-structure factor exposure is di erent in recessions and booms.

Figure: PCA Factor Loadings for Treasuries in Boom and Recession

(a) Level Factor (b) Slope Factor (c) Curvature Factor

This paper

Research Question:

1. Find latent factors and loadings that are state-dependent.
2. Test if factor model is state-dependent.

Key elements of estimator

1. Statistical factors instead of pre-speci ed (and potentially miss-speci ed) factors
2. Uses information from large panel data sets: Many cross-section units with many time observations
3. Factor structure can be time-varying as a general non-linear function of the state process

Contribution of this paper

Contribution

Theoretical

- PCA estimator combined with kernel projection for factors, state-varying factor loadings and common components
- Inferential theory for estimators for $N, T \to \infty$:
  - consistency
  - asymptotic normal distribution and standard errors
- Test for state-dependency of latent factor model
  - Generalized correlation test statistic detects for which states model changes
  - Non-standard superconsistency

Empirical

- State-dependency of factor loadings in US Treasury securities

Literature (partial list)

- Large-dimensional factor models with constant loadings
  - Bai (2003): Distribution theory
  - Fan et al. (2013): Sparse matrices in factor modeling
- Large-dimensional factor models with time-varying loadings
  - Su and Wang (2017): Local time-window
  - Pelger (2018), Aït-Sahalia and Xiu (2017): High-frequency
  - Fan et al. (2016): Projected PCA
- Large-dimensional factor models with structural breaks
  - Stock and Watson (2009): Inconsistency
  - Breitung and Eickmeier (2011), Chen et al. (2014): Detection

The Model

State-varying factor model

- X_{it} is the observed data for the i-th cross-section unit at time t
- State variable S_t at time t

$$X_{it} = \underbrace{\Lambda_i(S_t)^\top}_{\text{loadings}} \underbrace{F_t}_{\text{factors}} + \underbrace{e_{it}}_{\text{idiosyncratic}} \qquad i = 1, \ldots, N; \; t = 1, \ldots, T$$

  - N cross-section units (large), time horizon T (large)
  - r systematic factors (fixed)
- Factors F, loadings $\Lambda(S_t)$, idiosyncratic components e are unknown
- Data X and state process S_t observed

The Model

Examples (with one factor) equivalent to multi-factor representation

Loadings linear in state: ${}_i(S_t) = {}_{i;1} + {}_{i;2} S_t$

$$X_{it} = {}_{i;1} \underbrace{F_t}_{F_{t;1}} + {}_{i;2} \underbrace{(S_t F_t)}_{F_{t;2}} + e_{it}$$

Loadings nonlinear in discrete state: ${}_i(S_t) = g_i(S_t)$, $S_t \in \{s_1; s_2\}$

$$X_{it} = \underbrace{g_i(s_1)}_{i;1} \underbrace{1_{\{S_t = s_1\}} F_t}_{F_{t;1}} + \underbrace{g_i(s_2)}_{i;2} \underbrace{1_{\{S_t = s_2\}} F_t}_{F_{t;2}} + e_{it}$$

Our model

Loadings nonlinear in non-discrete state: ${}_i(S_t) = g_i(S_t)$ with continuous distribution function for S_t

$\Rightarrow$ Cumbersome/No multi-factor representation

The Model: Main Assumptions

Approximate state-varying factor model

- Systematic factors explain a large portion of the variance
- Idiosyncratic risk is nonsystematic: Weak time-series and cross-sectional correlation
- State: recurrent (in nite observations around the state to condition on) with continuous stationary PDF
- Factor Loadings: deterministic functions of the state and the functions are Lipschitz continuous (observations in the nearby state are useful)
  9C; k ¡(s + s) ¡(s)k Cj sj

The Model: Extension

Robustness to noise in state process

State process is observed with noise:

$$\underbrace{X_t}_{N\times 1} = \underbrace{\Lambda(S_t)}_{N\times r}\underbrace{F_t}_{r\times 1} + \underbrace{E_t}_{N\times r}\underbrace{F_t}_{r\times 1} + \underbrace{e_t}_{N\times 1} = \Lambda(S_t)F_t + \eta_t + e_t$$

Under weak conditions noise can be treated like idiosyncratic noise.

⇒ All results hold!

Missing relevant states

Assume loadings depend on multiple states but we only condition on a subset of them.

State-varying factor model explains strictly more variance than constant loading model.

⇒ More parsimonious representation even under misspecification.

The Model: Intuition

Intuition for Estimation

Constant loadings:
Loadings are principal components of covariance matrix

$$\mathrm{Cov}(X_t) = \quad \mathrm{Cov}(F_t) \quad ^\top + \mathrm{Cov}(e_t):$$

State-varying loadings:
Loadings for $S_t = s$ are principal components of covariance matrix conditioned on the state $S_t = s$:

$$\mathrm{Cov}(X_t | S_t = s) = (\quad s)\mathrm{Cov}(F_t | S_t = s) (\quad s)^\top + \mathrm{Cov}(e_t | S_t = s):$$

⇒ Intuition: Estimate conditional covariance matrix $\mathrm{Cov}(X_t | S_t = s)$ with kernel projection and apply PCA to it.

The Model: Nonparametric Estimation

Objective function and nonparametric estimation

The estimators minimize mean squared error conditioned on state:

$$\hat{F}^s, \hat{\Lambda}(s) = \arg\min_{F^s, \Lambda(s)} \frac{1}{NT(s)} \sum_{i=1}^{N} \sum_{t=1}^{T} K_s(S_t)\left(X_{it} - \Lambda_i(s)^\top F_t\right)^2$$

- Kernel function $K_s(S_t) = \frac{1}{h} K\left(\frac{S_t - s}{h}\right)$ (e.g. $K(u) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{u^2}{2}\}$)
- $T(s) = \sum_{t=1}^{T} K_s(S_t)$, $\frac{T(s)}{T} \xrightarrow{p} \pi(s)$ (stationary density of $S_t = s$)
- Bandwidth parameter h determines local "state window"

The Model: Nonparametric Estimation

Nonparametric estimation

Project square root of kernel on the data and factors

$$X_{it}^{s} = K_s^{1/2}(S_t) X_{it} \qquad F_t^{s} = K_s^{1/2}(S_t) F_t$$

PCA solves optimization problem

$$\hat{F}^{s}, \hat{\Lambda}(s) = \arg\min_{F^{s}, \Lambda(s)} \frac{1}{NT(s)} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(X_{it}^{s} - \Lambda_i(s)^{\prime} F_t^{s} \right)^2$$

⇒ Apply PCA to conditional covariance matrix

$\hat{F}^{s}$ are the eigenvectors corresponding to top k eigenvalues of estimated conditional covariance matrix $\frac{1}{NT(s)}(X^{s})^{\prime} X^{s}$

$\hat{\Lambda}(s)$ are coefficients from regressing X^{s} on $\hat{F}^{s}$

The Model: Nonparametric Estimation

Major challenge: Bias term

$$X_t^s = (S_t)F_t^s + e_t^s = \underbrace{(s)F_t^s + e_t^s}_{X_t^s} + \underbrace{((S_t)\ (s))F_t^s}_{X_t^s}:$$

$$X_{it}^s = {}_i(S_t)F_t^s \quad {}_i(s)F_t^s = O_p(h)$$

Kernel bias complicates problem and lowers convergence rates

Theorem: Consistency

Assume $N; Th \to \infty$ and ${}_{NT;h}h \to 0$ with ${}_{NT;h} = \min(\sqrt{N}; \sqrt{Th})$:

$$ {}^2_{NT;h} \frac{1}{T}\sum_{t=1}^{T} \hat{F}_t^s \quad (H^s)^T F_t^s \quad ^2 = O_p(1)$$

$$ {}^2_{NT;h} \frac{1}{N}\sum_{i=1}^{N} \hat{}_i(s) \quad (H^s)^{\ 1} {}_i(s) \quad ^2 = O_p(1)$$

for known full rank matrix H^s

Limiting Distribution of Estimated Factors

Theorem (Factors)

Assume $\sqrt{N}h/(Th) \to 0$, $Nh \to \infty$ and $Nh^2 \to 0$. Then

$$
\begin{aligned}
&\sqrt{N}\left(K_s^{-1/2}(S_t)\hat{F}_t^s - (H^s)'F_t\right)\\
&= (V_r^s)^{-1}\frac{(\hat{F}^s)'F^s}{T}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\Lambda_i(s)e_{it} + o_p(1)\\
&\xrightarrow{D} N\left(0, (V^s)^{-1}Q^s\Gamma_t^s(Q^s)'(V^s)^{-1}\right)
\end{aligned}
$$

- Rotation matrix $H^s = \frac{(\Lambda^s)'(\Lambda^s)}{N}\frac{(F^s)'\hat{F}^s}{T}(V_r^s)^{-1}$
- $K_s^{-1/2}(S_t)\hat{F}_t^s$ converges to some rotation of F_t at rate $\sqrt{N}$
- Efficiency mainly depends on asymptotic distribution of $\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\Lambda_i(s)e_{it}$

Limiting Distribution of Estimated Factor Loadings

Theorem (Loadings)

Assume $\sqrt{Th}/N \to 0$, $Th \to \infty$, and $Th^3 \to 0$. Then

$$\sqrt{Th}\left(\hat{\Lambda}_i(s) - (H^s)^{-1}\Lambda_i(s)\right)$$

$$= (V_r^s)^{-1}\frac{(\hat{F}^s)'F^s}{Th}\frac{(\Lambda^s)'(\Lambda^s)}{N}\frac{\sqrt{Th}}{T(s)}\sum_{t=1}^{T} F_t^s e_{it}^s + o_p(1)$$

$$\xrightarrow{D} N(0; ((Q^s)')^{-1}\Phi_i^s(Q^s)^{-1})$$

- $\hat{\Lambda}_i(s)$ converges to some rotation of $\Lambda_i(s)$ at rate $\sqrt{Th}$
- Efficiency mainly depends on asymptotic distribution of $\frac{\sqrt{Th}}{T(s)}\sum_{t=1}^{T} F_t^s e_{it}^s = \frac{\sqrt{Th}}{T(s)}\sum_{t=1}^{T} K_s(S_t)F_t e_{it}$

Limiting Distribution of Common Component

Theorem (Common Components)

Assume $Nh \to \infty$, $Th \to \infty$, $Nh^2 \to 0$ and $Th^3 \to 0$. Then for each i

$$\delta_{NT,h}(\hat{C}_{it,s} - C_{it,s}) = \frac{\delta_{NT,h}}{\sqrt{N}} \Lambda_i(s)' \Sigma_{\Lambda}^{-1}(s) \left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \Lambda_i(s) e_{it} \right) + \frac{\delta_{NT,h}}{\sqrt{Th}} F_t' \Sigma_{F|s}^{-1} \left(\frac{\sqrt{Th}}{T(s)} \sum_{t=1}^{T} F_t^s e_{it}^s \right) + o_p(1)$$

$$\delta_{NT,h} = \min(\sqrt{N}, \sqrt{Th})$$

Define $C_{it,s} = F_t^{0} \Lambda_i(s)$ and $\hat{C}_{it,s} = \left(\frac{\hat{F}_t^s}{K_s^{1/2}(S_t)} \right)' \hat{\Lambda}_i(s)$

If $N/(Th) \to 0$, $\Lambda_i(s) e_{it}$ dominates

If $Th/N \to 0$, $F^s(t) e_{it}^s$ dominates

Generalized Correlation

Test for constancy: Generalized correlation test

Consider loadings in two states $_1 = (s_1)$ and $_2 = (s_2)$. Test for

$$H_0: \; _1 = \; _2 G \text{ for some full rank square matrix } G$$
$$H_1: \; _1 \neq \; _2 G \text{ for any full rank square matrix } G$$

Generalized correlation, de ned as invariant of G

$$= \text{trace} \left(\left(\frac{^T_1 \; _1}{N} \right)^{1} \frac{^T_1 \; _2}{N} \left(\frac{^T_2 \; _2}{N} \right)^{1} \frac{^T_2 \; _1}{N} \right)$$

^ estimated and r is #factors

Equivalent to test $H_0: \; = r$ and $H_1: \; < r$

Generalized Correlation

Theorem: Generalized correlation test

Assume $\sqrt{N}/(Th) \to 0$, $Nh \to \infty$, $Th \to \infty$, $\sqrt{Th}/N \to 0$, $Nh^2 \to 0$ and $NTh^3 \to 0$:

$$\sqrt{NTh}(\hat{\ } \quad r \quad \hat{\ }^{\top} \hat{b}) \xrightarrow{d} N(0; \ )$$

- $^{\top} b$ bias term with feasible estimates $\hat{\ }$ and $\hat{\ }$
- feasible estimator for asymptotic covariance $\hat{\ }$
- ⇒ Superconsistent rate $\sqrt{NTh}$ (corner case)
- $h \in [1/T^{1/2}; 1/T^{3/4}]$: combinations of N and T exist to satisfy the rate conditions

Empirical Applications

US Treasury Securities Yields from 2001-07-31 to 2016-12-01:
N = 11, T = 2832: 1, 3, 6 mo., 1, 2, 3, 5, 7, 10, 20, 30 yr.

State: Log-normalized VIX

Generalized correlation: (‘(Boom); (Recession)) = 2 :6352
) reject 3 for (Boom) (Recession)

(a) Log-normalized VIX (b) Proportion of variance explained

Empirical Applications

Long term bonds have higher weights in the level factor in high VIX/recession

Figure: Factor Loading to the Level Factor (1st Factor)

(a) Log-normalized VIX

(b) Recession Indicator

Empirical Applications

In high vix/recession: short term bonds more negative and long term bonds less positive

Figure: Factor Loading to the Slope Factor (2nd Factor)

(a) Log-normalized VIX

(b) Recession Indicator

Empirical Applications

Minimum portfolio weight in the curvature factor shifts to shorter term bond in high vix/recession

Figure: Factor Loading to the Curvature Factor (3rd Factor)

(a) Log-normalized VIX

(b) Recession Indicator

Empirical Applications: Test Constancy of Loadings

- Loadings in low vix are di erent from loadings in high vix (red region)

Figure: Generalized Correlation Test of Estimated Loadings in Two States under Null Hypothesis (H_0: Loadings in Two States are Constant)

(a) t-value

(b) p-value

S&P500 Stock Returns

Daily stocks returns (01/2004 to 12/2016): $N = 332$ and $T = 3253$

State: Log-normalized VIX

) Constant loading model needs roughly three more factors to explain the same variation in- and out-of-sample.

Figure: Variation explained by state-varying and constant loading model.

S&P500 Stock Returns

Figure: Out-of-sample Sharpe ratio of mean-variance e cient portfolio based on latent factors of the state-varying and constant loading model.

) State-varying factor models capture more pricing information than constant-loading factors

Conclusion

Methodology

- Estimators for latent factors, loadings and common components where loadings are state-dependent
- We combine large dimensional factor modeling with nonparametric estimation
- Asymptotic properties of the estimators
- Constancy test for estimated state-varying factor loadings

Empirical Results

- We discover the movements of factor loadings by state values in the US Treasury Securities and Equity Markets
- Promising empirical results in other data sets

Data Generating Process for Simulations

- We generate data from a one-factor model

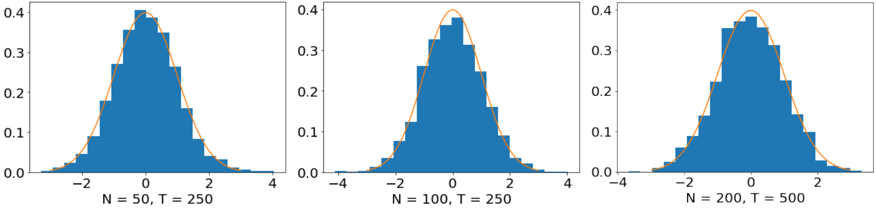
$$X_{it} = \Lambda_i(S_t)F_t + e_{it}$$

- Factor: $F_t \quad N(0; 1)$
- State: Ornstein–Uhlenbeck (OU) process (mean-reverting) $S_t = \ (\quad S_t)d_t + \ dW_t$, where $= 1$, $= 0{:}2$, and $= 1$
  stochastic volatility in financial data
- Loading: $\Lambda_i(S_t) = \Lambda_{0i} + \frac{1}{2}S_t\Lambda_{1i} + \frac{1}{4}S_t^2\Lambda_{2i} + \frac{1}{8}S_t^3\Lambda_{3i}$, where $\Lambda_{0i}; \Lambda_{1i}; \Lambda_{2i}; \Lambda_{3i} \quad N(0; 1)$
- Idiosyncratic errors: IID/Heteroskedasticity/Cross sectional dependence

Simulation of CLT for Estimated Factors

- $\sqrt{N}(\hat{\Gamma}_t^s)^{-1/2}(\hat{Q}^s)^{-1}\hat{V}^s\left(K_s^{-1/2}(S_t)\hat{F}_t^s - (H^s)'F_t\right) \xrightarrow{d} N(0; I_r)$

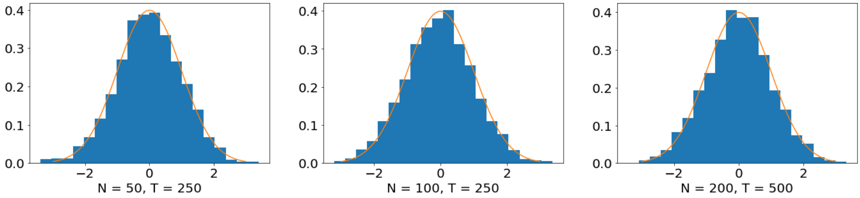
Figure: Comparison between simulated normalized factor distribution and standard normal distribution



Simulation of CLT for Estimated Loadings

- $\sqrt{Th}(\hat{\Phi}_i^s)^{-1/2}(\hat{Q}^s)'(\hat{\Lambda}_i(s) - (H^s)^{-1}\Lambda_i(s)) \xrightarrow{d} N(0, I_r)$

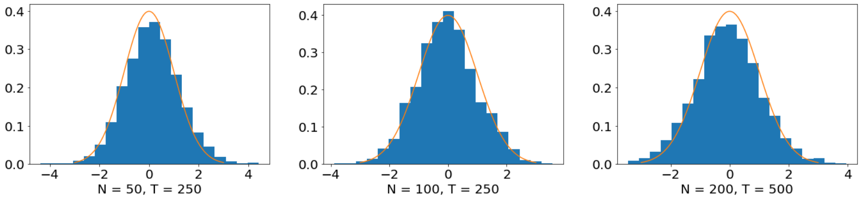
Figure: Comparison between simulated normalized loading distribution and standard normal distribution



Simulation of CLT for Common Component

- $\frac{1}{N}\hat{V}_{it;s} + \frac{1}{Th}\hat{W}_{it;s} \quad ^{1=2} \quad \hat{C}_{it;s} \quad C_{it;s} \quad \overset{d}{!} \; N(0;I_r)$

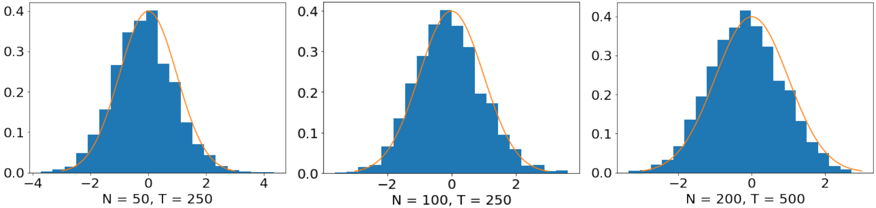
Figure: Comparison between simulated normalized common component distribution and standard normal distribution



Simulation of CLT for Estimated Generalized Correlation

- Loading: constant with the state $\Lambda_i(S_t) = \Lambda_{0i}$
- $\sqrt{NTh}(\hat{\rho} - \hat{\rho}^T \hat{b})/(\hat{\Omega})^{1/2} \xrightarrow{d} N(0;1)$

Figure: Comparison between simulated normalized estimated generalized correlation distribution and standard normal distribution



Recover Functional Form of Loadings vs. State

- $\Lambda_i(S_t) = \Lambda_{0i} + \frac{1}{2}S_t\Lambda_{1i} + \frac{1}{4}S_t^2\Lambda_{2i} + \frac{1}{8}S_t^3\Lambda_{3i}$

Figure: Loading as a function of the State