Interpretable Sparse Proximate Factors for Large Dimensions

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SoFiE 2019 Conference
June 13, 2019

Motivation: What are the factors?

Statistical Factor Analysis

- Factor models are widely used in big data settings
  - Summarize information and reduce data dimensionality
  - Problem: Which factors should be used?
- Statistical (latent) factors perform well
  - Factors estimated from Principle Component Analysis (PCA)
  - Weighted averages of all cross-section units
  - Problem: Hard to interpret

Goals of this paper:

Create interpretable sparse proximate factors

- Shrink most small factor weights to zero to get proximate factors

$\Rightarrow$ More interpretable!

Contribution of this paper

Contribution

- This Paper: Estimation of interpretable proximate factors
- Key elements of estimator:
  1. Statistical factors instead of pre-specified (and potentially miss-specified) factors
  2. Uses information from large panel data sets: Many cross section units with many time observations
  3. Proximate factors approximate latent factors very well with a few cross section units without sparse structure in population loadings
  4. Only 5-10% of the cross-sectional observations with the largest exposure are needed for proximate factors

Contribution

Theoretical Results

- Asymptotic probabilistic lower bound for generalized correlations of proximate factors with population factors
- Guidance on how to construct proximate factors

Empirical Results

- Very good approximation to population factors with 5-10% cross-section units, measured by generalized correlation and variance explained
- Interpret statistical latent factors for
  - 370 single-sorted anomaly portfolios
  - 128 macroeconomic variables

Literature (partial list)

- Large-dimensional factor models with PCA
  - Bai and Ng (2002): Number of factors
  - Bai (2003): Distribution theory
  - Fan et al. (2013): Sparse matrices in factor modeling
  - Fan et al. (2016): Projected PCA for time-varying loadings
  - Pelger (2019), Aït-Sahalia and Xiu (2017): High-frequency
  - Kelly, Pruitt and Su (2017): IPCA
- Factor models with penalty term
  - Bai and Ng (2017): Robust PCA with ridge shrinkage
  - Lettau and Pelger (2018): Risk-Premium PCA with pricing penalty
  - Zhou et al. (2006): Sparse PCA (low dimension)

Illustration (more details later...)

Portfolio Data

- Monthly return data from 07/1963 to 12/2016 ($T = 638$) for $N = 370$ portfolios
- Same data as in Lettau and Pelger (2018): 370 decile portfolios sorted according to 37 anomaly characteristics, e.g. momentum, volatility, turnover, size and volume,...
- Estimate a 5-factor model with PCA as in Lettau and Pelger (2018)
- Construct sparse factors with only 30 non-zero portfolio weights

⇒ 95% average correlation of proximate factors with PCA factors

⇒ Proximate factors explain 97% of the PCA variation, i.e. almost no loss in information

Characteristic Sorted Portfolios: Fourth Factor

- Hard to interpret...

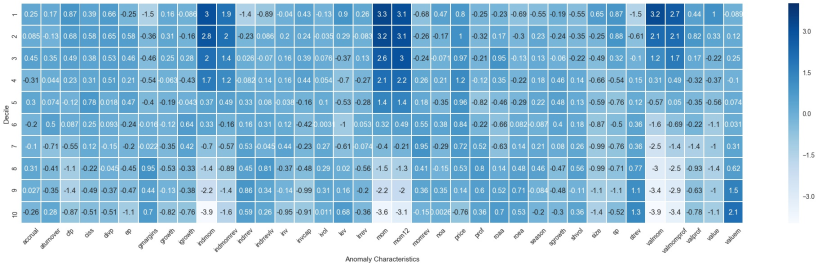


Figure: Financial single-sorted portfolios: Portfolio weights of 4th PCA factor.

Single-sorted Portfolios: Fourth Proximate Factor

- The fourth proximate factor is a long-short momentum factor

⇒ Long-short extreme portfolios sorted by Industry Mom. Reversals, Momentum (6m), Momentum (12m), Value-Momentum, Value-Momentum-Prof.

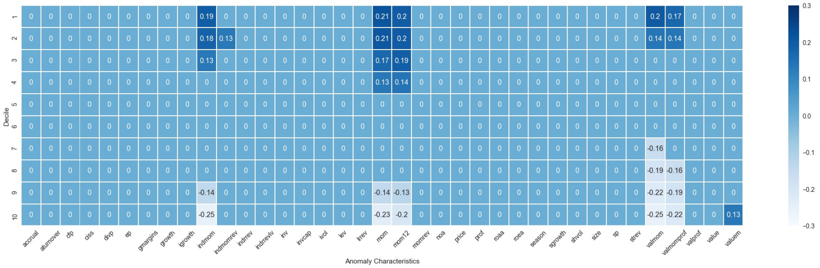


Figure: Portfolio weights of 4th proximate factor with 30 nonzero entries.

The Model

Approximate Factor Model

- Observe panel data of N cross-section units over T time periods:

$$X_{i,t} = \underbrace{\underset{1\times K}{\Lambda_i}^\top}_{loadings} \underbrace{\underset{K\times 1}{F_t}}_{factors} + \underbrace{e_{i,t}}_{idiosyncratic} \qquad i = 1, ..., N \quad t = 1, ..., T$$

- Matrix notation

$$\underbrace{X}_{N\times T} = \underbrace{\Lambda}_{N\times K} \underbrace{F^\top}_{K\times T} + \underbrace{e}_{N\times T}$$

  - N assets (large)
  - T time-series observation (large)
  - K systematic factors (fixed)
- F, Λ and e are unknown

The Model

Approximate Factor Model

- Systematic and non-systematic risk (F and e uncorrelated):

$$Var(X) = \underbrace{\Lambda Var(F)\Lambda^\top}_{systematic} + \underbrace{Var(e)}_{non-systematic}$$

$\Rightarrow$ Systematic factors explain a large portion of the variance
$\Rightarrow$ Idiosyncratic risk can be weakly correlated
$\Rightarrow$ Motivation for Principal Component Analysis!

Steps in Latent Factor Estimation

1. Estimate factor weights W (based on variation objective function)
2. Factors: $\hat{F} = X^\top W(W^\top W)^{-1}$
3. Loadings: $\hat{\Lambda} = X\hat{F}(\hat{F}^\top \hat{F})^{-1}$

$\Rightarrow$ Note that factor weights W do not need to coincide with loadings $\hat{\Lambda}$.

Estimation

Conventional PCA (Principal Component Analysis)

- PCA of sample covariance matrix $\frac{1}{T}XX^\top - \bar{X}\bar{X}^\top$.
- Eigenvectors of largest eigenvalues are weights and loadings $\hat{\Lambda} = W$.

Constructing Sparse Proximate Factors

- Estimate eigenvectors W by applying PCA to $\frac{1}{T}XX^\top - \bar{X}\bar{X}^\top$
- Sparse factor weights $\widetilde{W}_k$ are obtained from PCA weights W_k by
  - Keeping the m weights with largest absolute value for each k
  - Shrinking the rest to 0.
  - Dividing by column norm, i.e. $\tilde{W}_k^\top \tilde{W}_k = 1$
- Proximate factors $\tilde{F} = X^\top \widetilde{W}(\widetilde{W}^\top \widetilde{W})^{-1}$
- Loadings of proximate factors $\tilde{\Lambda} = X\tilde{F}(\tilde{F}^\top \tilde{F})^{-1}$

Closeness between Proximate Factors and Latent Factors

Closeness measure

- For 1-factor model: Correlation between $\tilde{F}$ and F.
- Challenge with multiple factors:
  - Factors only identified up to invertible linear transformations
  - Need measure for closeness between span of two vector spaces
- For multi-factor model: Measure distance between $\tilde{F}$ and F by generalized correlation.
  - Total generalized correlation measure:
    $$\rho = trace\left((F^T F/T)^{-1}(F^T \tilde{F}/T)(\tilde{F}^T \tilde{F}/T)^{-1}(\tilde{F}^T F/T)\right)$$
    - $\rho = 0$: $\tilde{F}$ and F are orthogonal
    - $\rho = K$: $\tilde{F}$ and F span the same space

Intuition: Why choose the largest PCA weights?

- Consider 1 factor and 1 nonzero element in $\widetilde{W}$: i.e. $K = 1$, $m = 1$.
- Note that PCA weights $W = \Lambda = [\lambda_{1,i}] \in \mathbb{R}^{N \times 1}$.
- Assume nonzero element in $\widetilde{W}_{1,i}$ is $\widetilde{W}_{1,1} = 1$.

$$\begin{aligned}\tilde{F} &= X^T \widetilde{W} = F\Lambda^T \widetilde{W} + e^T \widetilde{W} \\ &= f_1 \lambda_{1,1} + e_1\end{aligned}$$

- Assume

$$f_{1,t} \sim (0, \sigma_f^2), \qquad e_{1,t} \overset{iid}{\sim} (0, \sigma_e^2)$$
$$\frac{f_1^T f_1}{T} \to \sigma_f^2, \qquad \frac{e_1^T e_1}{T} \to \sigma_e^2$$

- Define signal-to-noise ratio $s = \frac{\sigma_f}{\sigma_e}$

Intuition: Why choose the largest PCA weights?

$$\begin{aligned}
\rho &= tr\left((F^TF/T)^{-1}(F^T\tilde{F}/T)(\tilde{F}^T\tilde{F}/T)^{-1}(\tilde{F}^TF/T)\right) \\
&= \left(\frac{f_1^T(f_1\lambda_{1,1}+e_1)/T}{(f_1^Tf_1/T)^{1/2}((f_1\lambda_{1,1}+e_1)^T(f_1\lambda_{1,1}+e_1)/T)^{1/2}}\right)^2 \\
&\to \frac{\lambda_{1,1}^2}{\lambda_{1,1}^2+1/s^2}
\end{aligned}$$

- (Generalized) correlation increases in size of loading $|\lambda_{1,1}|$.
- (Generalized) correlation increases in signal-to-noise ratio s.
- No sparsity in population loadings assumed!

$\Rightarrow$ We provide probabilistic lower bound for (generalized) correlation ρ given a target correlation level ρ_0:

$$P(\rho > \rho_0)$$

Intuition: Are Proximate Factors Consistent?

- Proximate factors $\tilde{F}$ are in general not consistent.
- Consider one-factor model

$$\tilde{F} = X^T \widetilde{W}(\widetilde{W}^\top \widetilde{W})^{-1} = F\Lambda^T \widetilde{W}(\widetilde{W}^\top \widetilde{W})^{-1} + e^T \widetilde{W}(\widetilde{W}^\top \widetilde{W})^{-1}$$

- Idiosyncratic component not diversified away
- Assume $e_{i,t} \overset{iid}{\sim} (0, \sigma_e^2)$, then $e^T \widetilde{W}$ satisfies

$$Var\left(\sum_{i=1}^{m} \widetilde{W}_{1,1_i} e_{1_i,t}\right) = \sigma_e^2 \nrightarrow 0$$

for fixed m.

Assumptions

Assumptions

Similar assumptions as in Bai and Ng (2002)

1. **Factors:** $E\left\|f_t\right\|^4 \leq M < \infty$ and $\frac{1}{T}\sum_{t=1}^{T} f_t f_t^T \xrightarrow{P} \Sigma_F$ for some $K \times K$ positive definite matrix $\Sigma_F = diag(\sigma^2_{f_1}, \sigma^2_{f_2}, \cdots, \sigma^2_{f_r})$.
2. **Loadings:** Random variables $\max_i \|\lambda_{j,i}\| = O_p(1)$ and $\Lambda^\top \Lambda / N \rightarrow \Sigma_\Lambda$, independent of factors and errors
3. **Systematic factors:** Eigenvalues of $\Sigma_\Lambda \Sigma_F$ bounded away from 0 and ∞
4. **Residuals:** Weak Dependency
   - Bounded eigenvalues and sparsity of Σ_e
   - e weakly dependent with F
   - Light tails

$\Rightarrow$ Uniform convergence result for loadings $\forall i$, $\exists H$,

$$\max_{i \leq N} \left\| \hat{\lambda}_{(i)} - H\lambda_{(i)} \right\| = O_p\left(\frac{1}{\sqrt{N}} + \frac{N^{1/4}}{\sqrt{T}}\right).$$

Loadings of Proximate Factors

Theorem 1: Consistency of loadings

The loadings of proximate factors converge to the population loadings:

$$\rho_{\tilde{\Lambda},\Lambda} \xrightarrow{P} K.$$

where $\rho_{\tilde{\Lambda},\Lambda}$ is the generalized correlation for the loadings:

$$\rho_{\tilde{\Lambda},\Lambda} = tr\left((\Lambda^\top \Lambda/N)^{-1}(\Lambda^\top \tilde{\Lambda}/N)(\tilde{\Lambda}^\top \tilde{\Lambda}/N)^{-1}(\tilde{\Lambda}^\top \Lambda/N)\right).$$

- Loadings span the same vector space
  $\Rightarrow$ same results in cross-sectional regressions, etc.
- Does not guarantee pointwise convergence

One Factor Case: Correlation of Proximate Factors

Theorem 2: Lower bound for correlation

Assume: $K = 1$ factor and there exists sequences of constants $\{a_{1,N} > 0\}$ and $\{b_{1,N}\}$ such that

$$P((|\lambda_{1,(1)}| - b_{1,N})/a_{1,N} \leq z) \to G_1(z),$$

Then for $N, T \to \infty$

$$P\left(\rho \geq \rho_0\right) \geq 1 - G_{1,m}(z) + o_p(1)$$
$$\rho_0 = \frac{\sigma_{f_1}^2 (a_{1,N} z + b_{1,N})^2}{\frac{1+h(m)}{m}\sigma_e^2 + \sigma_{f_1}^2 (a_{1,N} z + b_{1,N})^2}$$

G_1 is the Generalized Extreme Value (GEV) distribution function,

$$G_1 = exp\left\{-\left[1 + \xi\left(\frac{z-\mu}{\sigma}\right)\right]^{-1/\xi}\right\}$$

One Factor Case: Extreme value theory

A few examples for G_1 and $a_{1,N}$ and $b_{1,N}$ for $\lambda_{1,i}$:

1. $G_1 \sim$ Gumbel distribution:
   - Standard normal distribution ($\lambda_i \sim N(0,1)$): $a_{1,N} = \frac{1}{N\phi(b_{1,N})}$ and $b_{1,N} = \Phi^{-1}(1 - 1/N)$, where $\phi(\cdot), \Phi(\cdot)$ are pdf and cdf of standard normal.
   - Exponential distribution ($\lambda_i \sim exp(1)$): $a_{1,N} = 1, b_{1,N} = N$
2. $G_1 \sim$ Frechet distribution:
   - $F_\lambda(x) = exp(-1/x)$: $a_{1,N} = N, b_{1,N} = 0$.
3. $G_1 \sim$ Weibull distribution:
   - Uniform: distribution ($\lambda_i \sim Uniform(0,1)$): $a_{1,N} = 1/N, b_{1,N} = 1$.

$\Rightarrow$ allows $\lambda_{1,i}$ to be cross-sectionally dependent, characterized by an extremal index θ appearing in G_1

One Factor Case: Comparative Statics

For target probability $p = 1 - G_{1,m}(z)$, the threshold $\rho_0 = \frac{\sigma_{f_1}^2 (a_{1,N} z + b_{1,N})^2}{\frac{1+h(m)}{m}\sigma_e^2 + \sigma_{f_1}^2 (a_{1,N} z + b_{1,N})^2}$ s.t. $P(\rho \geq \rho_0) \geq p + o_p(1)$ satisfies

- ρ_0 increases in the signal-to-noise ratio $s = \sigma_{f_1} / \sigma_e$
- ρ_0 increases in the dispersion of loadings' distribution
- ρ_0 increases in $\#$ nonzeros m and N (from simulation)
- ρ_0 decreases in $h(m)$
  ($h(m)$ measures correlation in idiosyncratic errors)

Multi Factors

Challenges

- Thresholded weights/proximate factors are in general not orthogonal to each other
- Generalized correlation takes this into account

Additional Assumptions

1. Each cross section unit has only very large exposure to one factor
2. Tail distributions for each factor loading asymptotically independent

$\Rightarrow$ Needed only for theoretical derivation, but not for this approach to work in simulation and empirical applications

$\Rightarrow$ Assumptions can be relaxed: some cross section units have only large exposure to one factor after rotation by some matrix

Multi Factors

Theorem 3: Distribution of generalized correlation

The asymptotic lower bound equals

$$\lim_{N,T\to\infty} P\left(\rho \geq \rho_0\right) \geq \prod_{j=1}^{K} \left(1 - G^*_{j,m}(\tau)\right) - \lim_{N\to\infty} P(\sigma_{\min}(B) < \underline{\gamma}) \quad (1)$$

$$\rho_0 = K - \frac{(1+h(m))\sigma_e^2}{m\underline{\gamma}^2} \sum_{j=1}^{K} \frac{1}{s_j u^2_{j,N}(\tau)},$$

where $S = diag(s_1, s_2, \cdots, s_K)$ are the eigenvalues of $\Sigma_F \Sigma_\Lambda$ in decreasing order and $0 < \underline{\gamma} < 1$.

- $\Rightarrow$ $\prod_{j=1}^{K}\left(1 - G^*_{j,m}(\tau)\right)$: product of loadings' tail distributions (asymptotically independent)
- $\Rightarrow$ $B \propto S^{1/2}\Lambda^\top\tilde{\Lambda}$. $P(\sigma_{\min}(B) < \underline{\gamma})$: $\sigma_{\min}(B)$ measures how correlated one thresholded loading is to other population factor loadings

Characteristic Sorted Portfolios

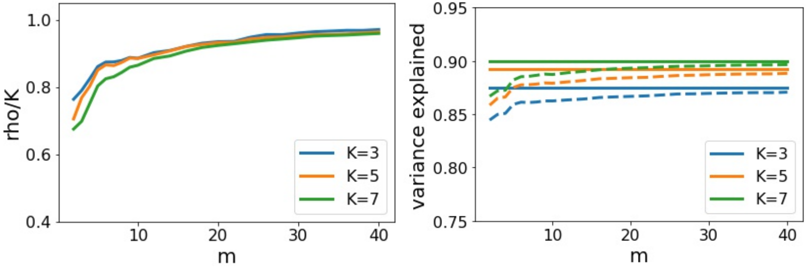
Portfolio Data (... continued)

- Monthly return data from 07/1963 to 12/2016 ($T = 638$) for $N = 370$ portfolios
- Same data as in Lettau and Pelger (2018): 370 decile portfolios sorted according to 37 anomaly characteristics, e.g. momentum, volatility, turnover, size and volume,...
- Estimate latent factors with PCA as in Lettau and Pelger (2018)
- Construct sparse factors with only $m = 30$ non-zero portfolio weights.

$\Rightarrow$ 95% Average correlation of proximate factors with PCA factors

$\Rightarrow$ Proximate factors explain 98% of the PCA variation, i.e. almost no loss in information

Characteristic Sorted Portfolios



(a) Generalized Correlation (b) Variance Explained

- Results for different number of factors K and sparsity levels m.
- Normalized generalized correlation ρ/K close to 1 implies same span

$\Rightarrow$ $m = 30$ achieves average correlation of 0.95%

$\Rightarrow$ $m = 30$ explains almost the same amount of variation as PCA.

Characteristic Sorted Portfolios

m	$\hat{F}_1$	$\hat{F}_2$	$\hat{F}_3$	$\hat{F}_4$	$\hat{F}_5$
10	0.993	0.992	0.771	0.918	0.837
20	0.995	0.948	0.883	0.949	0.890
30	0.996	0.965	0.935	0.966	0.910
40	0.997	0.971	0.958	0.975	0.923

Table: R^2 from regression of each PCA factor $\hat{F}_j$ on all proximate factors $\tilde{F}$ for $K = 5$.

- R^2 corresponds to generalized correlation between each $\hat{F}_j$ and all $\tilde{F}$.

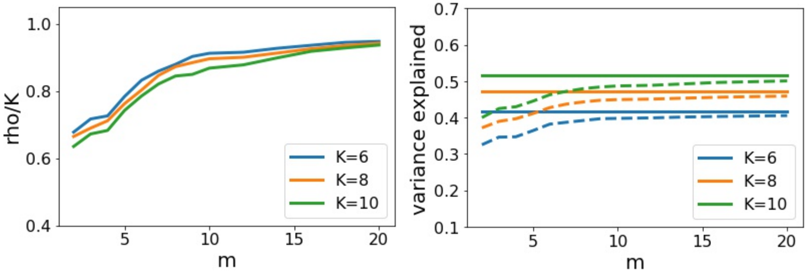
$\Rightarrow$ Proximate factors almost perfectly span the PCA factors with $m = 30$.

Macroeconomic data

Macroeconomic Data

- 128 Monthly U.S. macroeconomic indicators from from 01/1959 to 02/2018 from McCracken and Ng (2016): $N = 128$ and $T = 707$
- McCracken and Ng (2016) suggest $K = 8$ factor model.
- 8 different categories:
  1. output and income
  2. labor market
  3. housing
  4. consumption, orders and inventories
  5. money and credit
  6. interest and exchange rates
  7. prices
  8. stock market

Macroeconomic Data



(a) Generalized Correlation (b) Variance Explained

- Results for different number of factors K and sparsity levels m.
- Normalized generalized correlation ρ/K close to 1 implies same span

$\Rightarrow$ $m = 10$ achieves average correlation of 0.95%

$\Rightarrow$ $m = 10$ explains almost the same amount of variation as PCA.

Macroeconomic Data

m	$\hat{F}_1$	$\hat{F}_2$	$\hat{F}_3$	$\hat{F}_4$	$\hat{F}_5$	$\hat{F}_6$	$\hat{F}_7$	$\hat{F}_8$
10	0.953	0.959	0.949	0.953	0.961	0.799	0.833	0.767
15	0.967	0.970	0.958	0.956	0.964	0.857	0.867	0.837
20	0.977	0.974	0.957	0.963	0.961	0.905	0.919	0.891
25	0.983	0.980	0.961	0.979	0.973	0.937	0.943	0.929

Table: R^2 from regression of each PCA factor $\hat{F}_j$ on all proximate factors $\tilde{F}$ for $K = 8$.

- R^2 corresponds to generalized correlation between each $\hat{F}_j$ and all $\tilde{F}$.

$\Rightarrow$ Proximate factors closely span the PCA factors with $m = 10$.

Macroeconomic Data: Interpretation of Factors

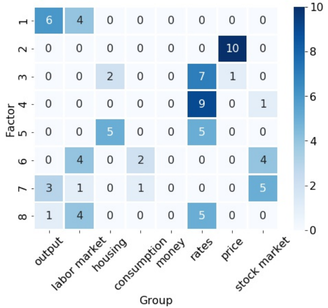


Figure: Non-zero weights by group for $K = 8$ factors and $m = 10$ non-zero entries.

- Proximate factors have clear patterns in weights.
- Interpretation of factors: (1) Productivity, (2) Price, (3) Interest, (4) Exchange-Rate, (5) Housing, (6)Finance/Labor, (7) Finance/Productivity, (8) Labor/Rates

Conclusion

Methodology

- Proximate factors (portfolios of a few cross-section units) for latent population factors (portfolios of all cross-section units)
- Simple thresholding estimator based on largest loadings
- Proximate factors approximate population factors well without sparsity assumption
- Asymptotic probabilistic lower bound for (generalized) correlation

$\Rightarrow$ A few observations summarize most of the information

Empirical Results

- Good approximation to population factors with 5-10% cross-section units

Relationship with Lasso: Sparse PCA

Alternative approach with Lasso:

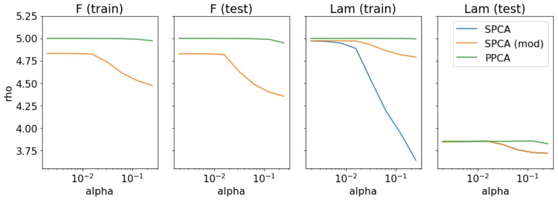
1. Estimate factors by PCA, i.e $X^T X \hat{F} = \hat{F} V$ with V matrix of eigenvalues.
2. Estimate loadings by minimizing $\left\| X - \Lambda \hat{F}^T \right\|_F^2 + \alpha \|\Lambda\|_1$. Divide the minimizer by its column norm (standardize each loading) to obtain $\bar{\Lambda}$
3. Proximate factors from Lasso approach are $\bar{F} = X^T \bar{\Lambda}(\bar{\Lambda}^T \bar{\Lambda})^{-1}$

⇒ Same selection of non-zero elements (for one factor case) but different weighting

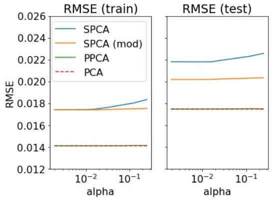
⇒ Under certain conditions worse performance than thresholding approach

- Tuning parameter less transparent
- Note that conventional sparse PCA assumes sparse loadings Λ and sparse factor weights W and sets them equal.

Characteristic Sorted Portfolios: Sparse PCA



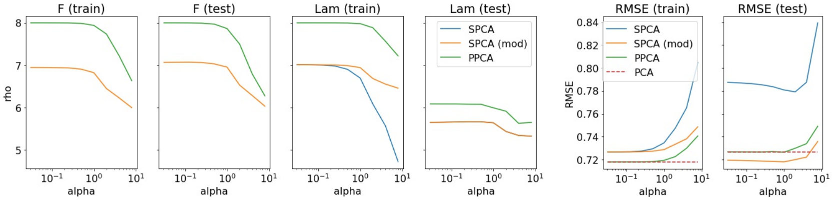
(a) Generalized correlations



(b) RMSE

Figure: Generalized correlations for factors and loadings and RMSE for proximate PCA (PPCA), sparse PCA (SPCA) and modified sparse PCA with second stage loading regression. α is the ℓ_1 penalty for SPCA with m chosen accordingly.

Macroeconomic data: Sparse PCA



(a) Generalized correlations (b) RMSE

Figure: Macroeconomic data: Generalized correlations for factors and loadings and RMSE for proximate PCA (PPCA), sparse PCA (SPCA) and modified sparse PCA with second stage loading regression. α is the ℓ_1 penalty for SPCA with m chosen accordingly.

Multiple Factors

Multiple Factor: Rotate and threshold

- Assume there exists orthonormal matrix P s.t. large values in columns of $W^P = \Lambda HSP$ do not overlap (almost orthogonal)
- m nonzero entries in $\tilde{W}_j$ are the largest in $\hat{W}_j$ satisfying $\max_{j,k\neq j} |\hat{w}^P_{i,k}/\hat{w}^P_{i,j}| < c$ and are standardized by

$$\tilde{W}^P = \left[\frac{\hat{W}^P_1 \odot M_1}{\|\hat{W}^P_1 \odot M_1\|} \quad \frac{\hat{W}^P_2 \odot M_2}{\|\hat{W}^P_2 \odot M_2\|} \quad \cdots \quad \frac{\hat{W}^P_K \odot M_K}{\|\hat{W}^P_K \odot M_K\|} \right].$$

- The proximate factors are

$$\tilde{F}^P = X^T \tilde{W}^P ((\tilde{W}^P)^T \tilde{W}^P)^{-1} = X^T \tilde{W}^P$$

- Generalized Correlation

$$\rho = tr\left((F^T F/T)^{-1} (F^T \tilde{F}^P/T)((\tilde{F}^P)^T \tilde{F}^P/T)^{-1}((\tilde{F}^P)^T F/T) \right)$$

Multiple Factors

Theorem 4: Rotate and threshold

Let $\bar{w}^P_{(m),j}$ be the m-th order statistic of the entries in $|w^P_j|$ that satisfy $\max_{j,k \neq j} |w^P_{i,k}/w^P_{i,j}| < c$ and assume that the cumulative density function of $\bar{w}^P_{(m),j}$ is continuous. Then for a particular threshold $0 < \rho_0 < K$ and a fixed m, we have

$$\lim_{N,T\to\infty} P(\rho > \rho_0) \geq \lim_{N\to\infty} P\left(\sum_{j=1}^{K} \frac{1}{(\bar{w}^P_{(m),j})^2} < \frac{m(1-\gamma)(K-\rho_0)}{(1+f(m))\sigma_e^2}\right), \qquad (2)$$

where $\gamma = c(2 + c(K-2))(K(K-1))^{1/2}$.

Simulation

- Compare probabilistic lower bounds with Monte-Carlo simulations
- **Factors:** $K = 1$ or $K = 2$ and $F_t \sim N(0, \sigma_f^2)$
- **Loadings:** $\lambda_i \sim N(0, 1)$ i.i.d.
- **Residuals:** $\sigma_e = 1$ and $e_{t,i} \sim N(0, 1)$ i.i.d.
- Vary signal-to-noise ratio with $\sigma_f \in \{0.8, 1.0, 1.2\}$
- $N = 100$) and $T \in \{50, 100, 200\}$
- We analyze:
  - Probabilistic lower bound for $\rho_0 = 0.95$
  - Distribution of lower bound with extreme value distribution

Simulation: One factor with very strong signal

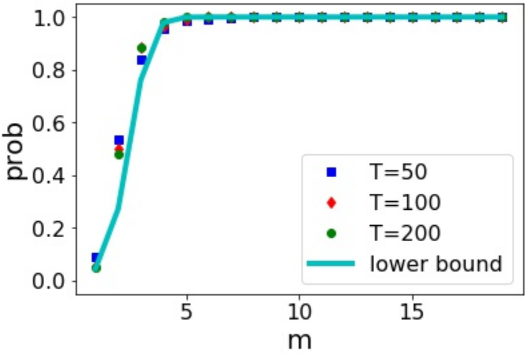


Figure: Probabilistic lower bound: $\sigma_f = 1.2$, $\rho_0 = 0.95$

Simulation: One factor with weaker signal

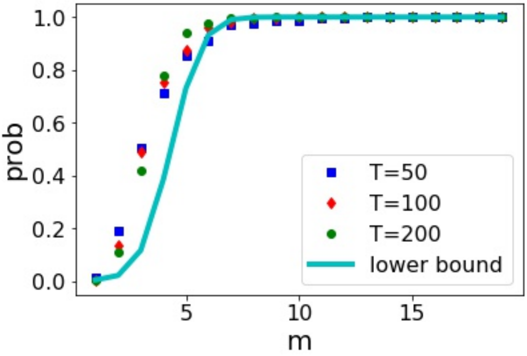


Figure: Probabilistic lower bound: $\sigma_f = 1.0$, $\rho_0 = 0.95$

Simulation: One factor with weak signal

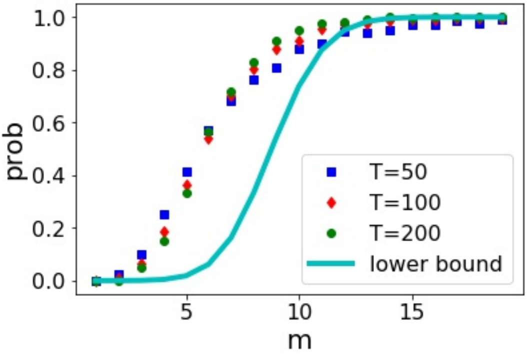
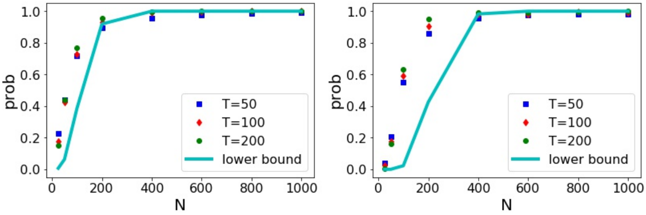


Figure: Probabilistic lower bound: $\sigma_f = 0.8$, $\rho_0 = 0.95$

Simulation: One factor with increasing N

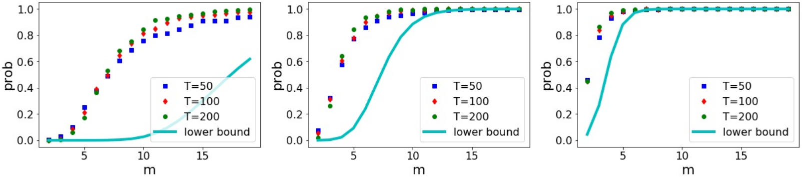


(a) One-factor model ($\sigma_f = 1.0$)

(b) Multi-factor model ($\sigma_f = [1.2, 1.0]$)

Figure: Probabilistic lower bound: $\rho_0 = 0.95$

Simulation: Two Factors

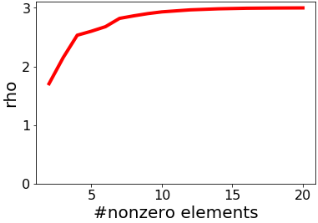


(a) $\sigma_f = [1.0, 0.8]$, (b) $\sigma_f = [1.2, 1.0]$ (c) $\sigma_f = [1.5, 1.2]$

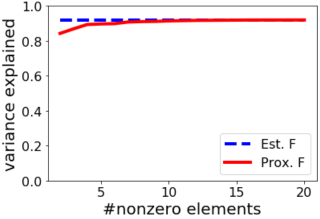
Figure: Probabilistic lower bound: $\rho_0 = 1.9$.

Empirical Application: Size and Investment Portfolios

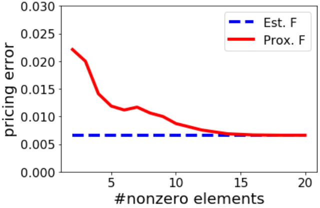
- 25 portfolios formed on size and investment (07/1963-10/2017, 3 factors, daily data)



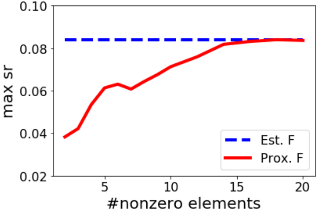
(a) Generalized correlation



(b) Variance explained



(c) RMS pricing error



(d) Max Sharpe Ratio

Empirical Application: Size and Investment Portfolios

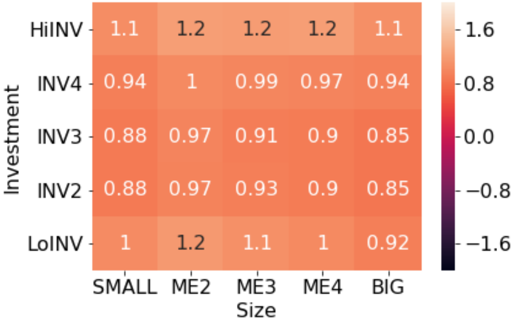


Figure: Portfolio weights of 1. statistical factor

⇒ Equally weighted market factor

Empirical Application: Size and Investment Portfolios

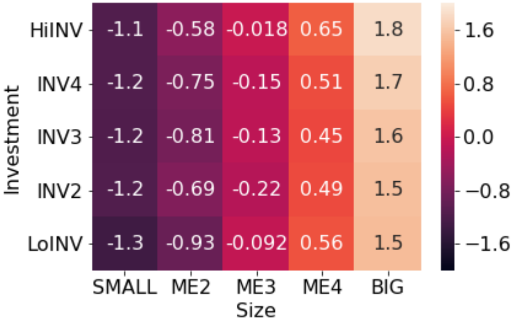


Figure: Portfolio weights of 2. statistical factor

⇒ Small-minus-big size factor

⇒ Proximate factor with 4 largest weights correlation 0.97 with size factor

Empirical Application: Size and Investment Portfolios

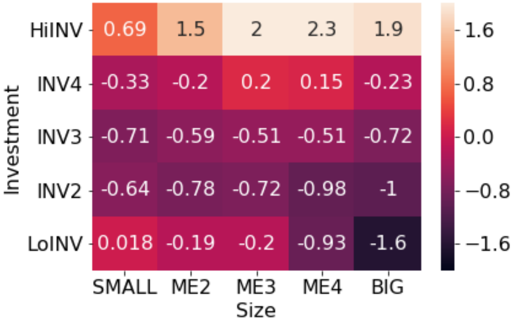


Figure: Portfolio weights of 3. statistical factor

⇒ High-minus-low value factor

⇒ Proximate factor with 4 largest weights correlation 0.79 with investment factor

Single-sorted Portfolios: First Proximate Factor

- The first proximate factor is a market factor.

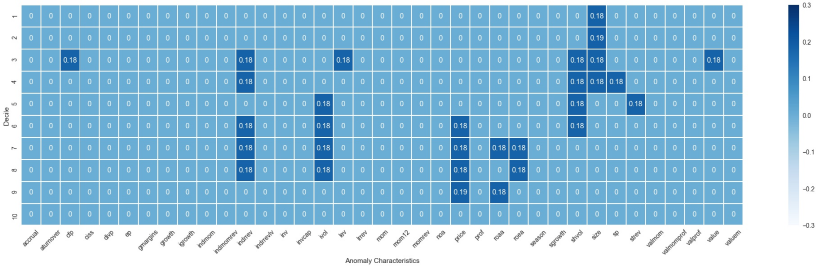


Figure: Portfolio weights of 1st proximate factor with 30 nonzero entries.

Single-sorted Portfolios: Second Proximate Factor

- The second proximate factor has large (in absolute value) loadings of value/growth related portfolios.

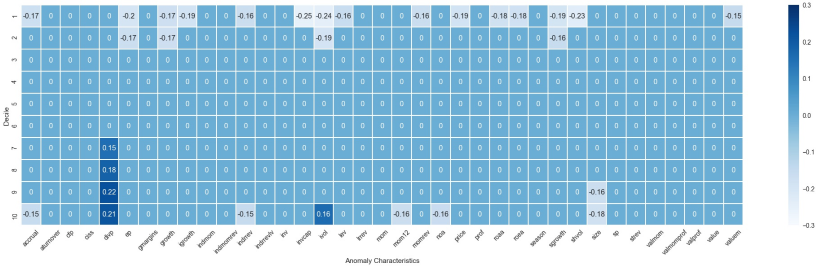


Figure: Portfolio weights of 2nd proximate factor with 30 nonzero entries.

Single-sorted Portfolios: Third Proximate Factor

- The third proximate factor loads most on momentum and profitability-related portfolios.

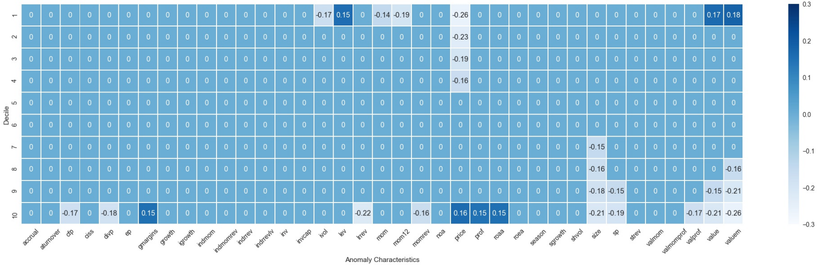


Figure: Portfolio weights of 3rd proximate factor with 30 nonzero entries.

Single-sorted Portfolios: Fifth Proximate Factor

- The fifth proximate factor a “high SR” factor.

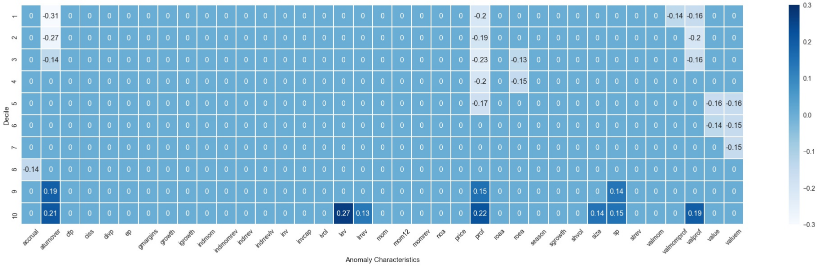


Figure: Portfolio weights of 5th proximate factor with 30 nonzero entries.

Single-sorted portfolios

	Anomaly characteristics		Anomaly characteristics
1	Accruals - accrual	20	Momentum (12m) - mom12
2	Asset Turnover - aturnover	21	Momentum-Reversals - momrev
3	Cash Flows/Price - cfp	22	Net Operating Assets - noa
4	Composite Issuance - ciss	23	Price - price
5	Dividend/Price - divp	24	Gross Protability - prof
6	Earnings/Price - ep	25	Return on Assets (A) - roaa
7	Gross Margins - gmargins	26	Return on Book Equity (A) - roea
8	Asset Growth - growth	27	Seasonality - season
9	Investment Growth - igrowth	28	Sales Growth - sgrowth
10	Industry Momentum - indmom	29	Share Volume - shvol
11	Industry Mom. Reversals - indmomrev	30	Size - size
12	Industry Rel. Reversals - indrrev	31	Sales/Price - sp
13	Industry Rel. Rev. (L.V.) - indrrevlv	32	Short-Term Reversals - strev
14	Investment/Assets - inv	33	Value-Momentum - valmom
15	Investment/Capital - invcap	34	Value-Momentum-Prof. - valmomprof
16	Idiosyncratic Volatility - ivol	35	Value-Protability -valprof
17	Leverage - lev	36	Value (A) - value
18	Long Run Reversals - lrrev	37	Value (M) - valuem
19	Momentum (6m) - mom		