

# Optimal Experimental Design for Staggered Rollouts

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## Introduction

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## Designing experiments with staggered rollouts

- Estimating treatment effects in panel data with **staggered rollouts**
  - Units  $i \in \{1, \dots, N\}$ ,  $N$  observed in time periods  $s \in \{1, \dots, T\}$
  - Design: Treatment assignment  $Z_{is} \in \{0, 1\}$
  - Potential outcomes:  $Y_{is}(z_{i,s-\ell}, \dots, z_{is})$  may depend on the history of treatment to date, with known  $\ell$  periods of history that matter
  - Observed outcomes:  $Y_{is} = Y_{is}(Z_{i,s-\ell}, \dots, Z_{is})$
- Staggered rollout designs commonly encountered in observational data:
  - Products/promotions released in different regions at different times
  - State regulations adopted over time
- Question: How should analyst **design** a staggered rollout experiment?
  - How fast should rollout occur?
  - How does rollout depend on hypothesized maximum duration of carryover effects?
  - How can historical data be used to optimize design?
  - Can an **adaptive design**, where analyst updates speed of rollout and termination based on data collected during experiment, improve performance?

## Panel experiments with staggered rollouts

**Formal objective:** Propose experimental designs that optimize the **precision** of post-experiment estimates of treatment effects

**Focus on environment with:** **Irreversible** treatment adoption pattern

( $Z_{is}$   $Z_{i,s+1}$ )

	Time		
SF	1	1	1
BOS	0	1	1
ATL	0	0	1
⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮

0 denotes control and 1 denotes treated

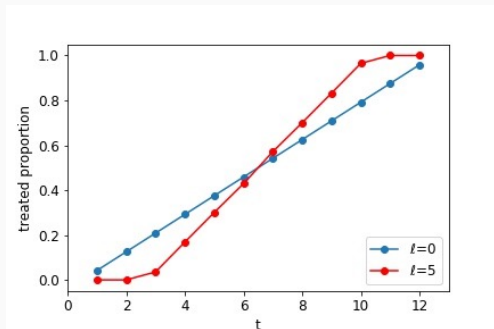
## Contribution: Non-adaptive experiments

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**Non-adaptive experiments:**  $N$  and  $T$  are set, and treatment decisions are made, **pre-experiment**

- Assume after experiment will use GLS to estimate **instantaneous and lagged treatment effects** from **nonstationary** observed outcomes
- **Analytical optimality conditions** for the designs that maximize linearly combined precisions of estimated instantaneous and lagged effects
- Propose an **algorithm** to choose a treatment design based on the optimality conditions. The design has **two** features
  - ) Fraction of treated units per period takes an **S-shaped curve**: Treatment **rollouts slowly** at the **beginning** and **end**, and **quickly** in the **middle**
  - Bigger  $\ell$  leads to more pronounced **S**
  - ) This rollout pattern is imposed for **each stratum** of units with the same **observed** and **estimated latent covariate** values

# Illustration of optimal assignment



## Contribution: Adaptive experiments

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**Adaptive experiments:**  $N$  is fixed, but the experiment can be terminated early. Treatment decisions are updated after each period's data is collected

- Propose the Precision-Guided Adaptive Experiment (PGAE) algorithm
  - adaptively terminates the experiment based on the estimated precision
  - adaptively optimizes speed of rollout using dynamic programming
  - an estimation scheme of treatment effects based on sample splitting
- Derive the asymptotic normal distribution of final treatment effect and variance estimates from PGAE
  - Optimal convergence rate and no efficiency loss of final treatment effect estimate, as compared to an oracle with access to the same design a-priori

## Related literature (partial list)

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- Most closely related to **stepped wedge designs** in clinical trials (Hussey and Hughes 2007, Hemming et al. 2015, Li, Turner, and Preisser 2018)
  - ) We study the design under a **more general outcome specification**, where **cumulative effects** can **vary with treatment duration**
- Recently proposed alternative designs for **estimation of carryover effects**
  - **Minimax temporal experimental design** (Basse, Ding, and Toulis 2019)
  - **Switchback design** (Bojinov, Simchi-Levi, and Zhao 2020)
  - **Synthetic control design** that selects units for (simultaneous) treatment, anticipating synthetic control estimation (Doudchenko et al. 2021a,b, Abadie and Zhao 2021)
  - ) Our design leverages **variation of treatment times across units** and **maximizes the precision** of treatment effect estimates
- Recently proposed designs in settings with **interference**
  - **Multiple randomization designs** (Bajari et al. 2021, Johari et al. 2022)
  - **Equilibrium designs** (Wager and Xu 2021)
  - ) Our experiment is run at the **aggregate** level and leverages the **time dimension** to increase power
- ) We also consider **adaptive** designs; above papers pre-specify design



## Two examples for staggered rollout experiments

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**Example 1 (marketplace experiments):** A ride-hailing platform plans to test the impact of a **new app feature** that improves **driver experience**

**Example 2 (public health intervention):** A country aims to measure the effect of a **new public health intervention** (e.g., encouraging the use of masks or social distancing policies) on the **spread of an infectious disease**

**Staggered rollout experiments** run at the city level for multiple time periods can

- avoid bias from interference
- facilitate the estimation of cumulative effects
- **better design** can **improve** the estimation **precision** of cumulative effects

## Setup

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## Potential outcomes and treatment effects

- The potential outcomes for unit  $i$  at time  $s$  can be written as

$$Y_{is}(Z_{i,s-\ell}, \dots, Z_{i,s-1}, Z_{is})$$

for a nonnegative, known integer  $\ell$  ( $\ell$ : duration of treatment effects)

- Let the average instantaneous effect  $\tau_0$  and  $j$ -th period lagged effect  $\tau_j$  be

$$\tau_j := \frac{1}{NT} \sum_{i,s} \left[ Y_{is}(0, \dots, 0, \underbrace{1}_{Z_{i,s-j}}, 1, \dots, 1) - Y_{is}(0, \dots, 0, \underbrace{0}_{Z_{i,s-j}}, 1, \dots, 1) \right],$$

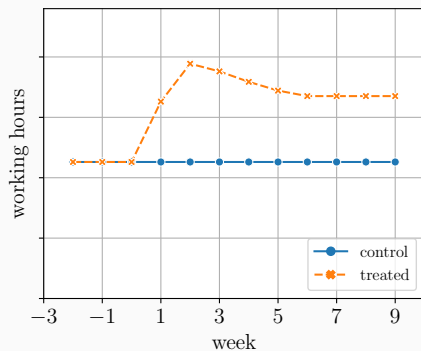
for all  $j \in \{0, 1, \dots, \ell\}$

- Let the average cumulative effect of treatment for  $j$  periods be

$$\tau_0 + \tau_1 + \dots + \tau_{\min(\ell, j-1)}$$

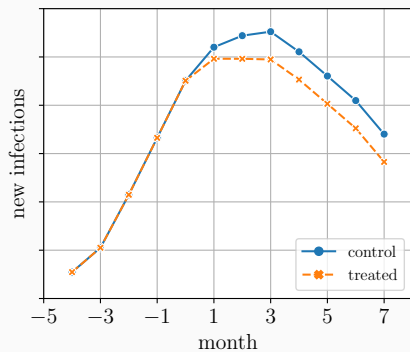
that is constant for  $j > \ell$

## Illustrative examples of cumulative effects



Cumulative effect of treatment for  $j$  periods with  $\ell = 5$ ,  $\tau_0, \tau_1 > 0$  and  $\tau_2, \tau_3, \tau_4, \tau_5 < 0$

## Illustrative examples of cumulative effects



Cumulative effect of treatment for  $j$  periods with  $\ell = 2$  and  $\tau_0, \tau_1, \tau_2 < 0$

## Outcome specification

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A general outcome specification for treatment effect estimation post-experiment

$$Y_{is} = \alpha_i + \beta_s + \mathbf{X}_i^\top \boldsymbol{\theta}_s + \tau_0 z_{is} + \tau_1 z_{i,s-1} + \dots + \tau_\ell z_{i,s-\ell} + \underbrace{\mathbf{u}_i^\top \mathbf{v}_s}_{e_{is}} + \varepsilon_{is}$$

- $\alpha_i$ : unknown unit fixed effect
- $\beta_s$ : unknown time fixed effect
- $\mathbf{X}_i$ : observed covariates;  $\boldsymbol{\theta}_s$ : unknown time-varying coefficients
- $\mathbf{u}_i$ : latent covariates;  $\mathbf{v}_s$ : latent coefficients
- $\varepsilon_{is}$ : iid residual with mean 0 and variance  $\sigma^2$

## Decision making problem

**Decision:** Optimally choose the **treatment times** for each unit

**Goal:** Most precisely estimate average instantaneous and lagged effects

**Implication:** Reduce sample size requirement and lower the experimental cost!

	Time		
SF	1	1	1
BOS	0	1	1
ATL	0	0	1
⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮

0 denotes control and 1 denotes treated

0	0
1	1

 $Z_{ff}?$ 

0	1
0	1

 $Z_{ba}?$ 

0	0
0	1

 $Z_{ffba}?$

## **Non-adaptive experiments**

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## Estimation of average treatment effects

GLS estimator  $\hat{\tau}_0, \hat{\tau}_\ell$  from the specification

$$Y_{is} = \alpha_i + \beta_s + \mathbf{X}_i^\top \boldsymbol{\theta}_s + \tau_0 z_{is} + \tau_1 z_{i,s-1} + \dots + \tau_\ell z_{i,s-\ell} + \mathbf{e}_{is},$$

- GLS is the best linear unbiased estimator (BLUE)
- Precision matrix (inverse of variance-covariance matrix) of  $\hat{\tau}_0, \hat{\tau}_\ell$ , denoted by  $\text{Prec}(\hat{\tau}_0, \hat{\tau}_\ell; \mathbf{Z})$ , is a quadratic function of  $\mathbf{Z} = [z_{is}]_{(i,s) \in [M] \times [T]}$ , where  $[M]$  stands for  $\{1, 2, \dots, Ng\}$

## Designs of non-adaptive experiments

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Trace(**T**)-optimal design: Choose  $Z = [z_{is}]_{(i,s) \in [M] \times [T]}$  pre-experiment to maximize the trace of the precision matrix (Pukelsheim, 2016)

$$\begin{aligned} \max_Z \quad & \text{trace}(\text{Prec}(\hat{\tau}_0, \hat{\tau}_\ell; Z)) \\ \text{s.t.} \quad & z_{is} \in \{0, 1\} \end{aligned}$$

Other objective functions, for example, determinant(**D**)-optimal design and **A**-optimal design

- No analytical solutions in general
- Numerical solutions for **D**-optimal design in the paper

## Optimal solution (No covariates)

$$Y_{is} = \alpha_i + \beta_s + \tau_0 Z_{is} + \tau_1 Z_{i,s-1} + \dots + \tau_\ell Z_{i,s-\ell} + \varepsilon_{is} \quad (1)$$

### Theorem 1: Optimal solution (no covariates)

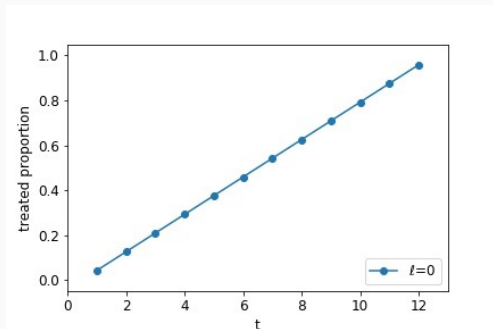
Under the specification (1),  $\varepsilon_{is}$  *i.i.d.*  $(0, \sigma^2)$  and  $\tau_j$  is estimated from OLS. Then any treatment design is optimal if it satisfies

$$\omega_s = \frac{1}{N} \sum_i Z_{is} = \omega_{\ell,s}^* .$$

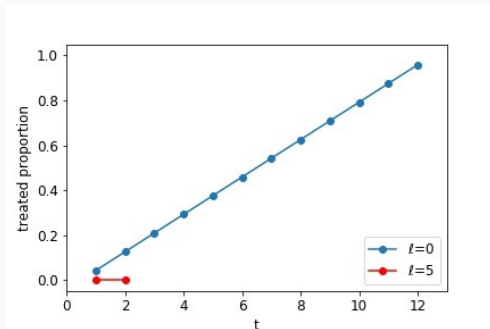
If  $\ell = 0$ , then  $\omega_{\ell,s}^* = (2s - 1)/(2T)$ .

For general  $\ell$ ,  $\omega_{\ell,s}^*$  has five stages, and the expression of  $\omega_{\ell,s}^*$  is provided in the paper.

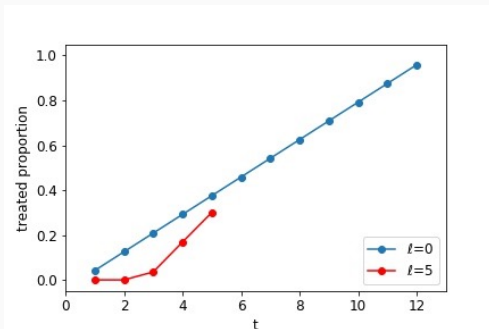
## Visualization of $\omega_{l,s}$ in optimal solution



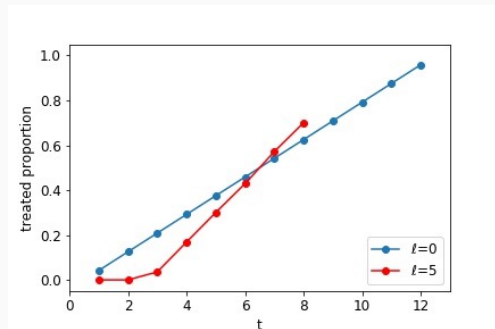
## Visualization of $\omega_{l,s}$ in optimal solution



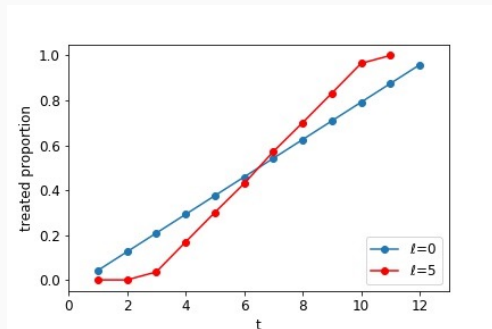
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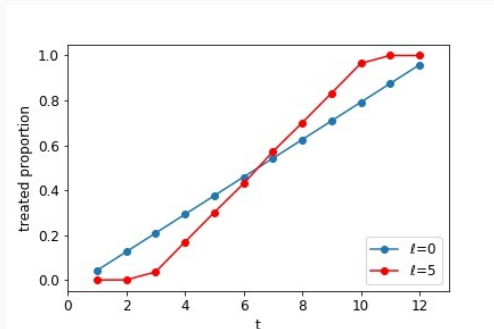


## Visualization of $\omega_{l,s}$ in optimal solution





## Visualization of $\omega_{l,s}$ in optimal solution



## Optimal solution (Adding observed and/or latent covariates)

$$Y_{is} = \alpha_i + \beta_s + \mathbf{X}_i^\top \boldsymbol{\theta}_s + \tau_0 Z_{is} + \tau_1 Z_{i,s-1} + \dots + \tau_\ell Z_{i,s-\ell} + \underbrace{\mathbf{u}_i^\top \mathbf{v}_s}_{e_{is}} + \varepsilon_{is} \quad (2)$$

### Theorem 1: Optimal solution (with covariates)

Under the specification (2),  $\varepsilon_{is} \stackrel{i.i.d.}{\sim} (0, \sigma^2)$ , both  $\mathbf{X}_i$  and  $\mathbf{u}_i$  are demeaned, and  $\tau_j$  is estimated from infeasible GLS. Then any treatment design is optimal if it satisfies

- $\omega_s = N^{-1} \sum_i Z_{is} = \omega_{\ell,s}^*$
- $N^{-1} \sum_i \mathbf{X}_i Z_{is}$  is fixed for all  $s$
- $N^{-1} \sum_i \mathbf{u}_i Z_{is}$  is fixed for all  $s$

## Interpretation of optimal solution (with covariates)

With  $\mathbf{X}_i$  only: Stratification if  $\mathbf{X}_i$  is discrete-valued

- Each stratum (group of units with the same  $\mathbf{X}_i$ ) satisfies the treated fraction conditions  $\omega_{\ell,s}^*$  (possibly with rounding)

With  $\mathbf{u}_i$ :  $\mathbf{u}_i$  is unknown in practice

- Estimate  $\mathbf{u}_i$  using historical data
- Partition units into strata based on  $\hat{\mathbf{u}}_i$

An algorithm proposed in the paper to choose a treatment design

$$\mathbf{X}_i = \mathbf{x}_1 \left\{ \begin{array}{cc} 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{array} \right.$$

$$\mathbf{X}_i = \mathbf{x}_2 \left\{ \begin{array}{cc} 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{array} \right.$$

## **Adaptive experiments**

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## Decisions for adaptive experiments

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Goal: Most precisely estimate average treatment effects with valid inference, using the least sample size

Two adaptive decisions:

- Stop the experiment early if the desired precision is achieved (i.e., max duration is  $T_{\max}$ , and duration  $\tilde{T} \in [T_{\max}]$  is a random variable)
- Speed of treatment rollout for the next time period is determined after each period's outcomes are collected

This talk: Focus on a simpler specification

$$Y_{is} = \alpha_j + \beta_s + \tau_0 z_{is} + \varepsilon_{is}$$

## Decision 1: Experiment termination rule

Terminate the experiment if the precision exceeds a target threshold  $c$  at time  $t$  (Glynn and White 1992)

$$\text{Prec}(\hat{\tau}_0; Z) \geq c$$

where  $Z \geq f_0, 1g^{N \times t}$  and

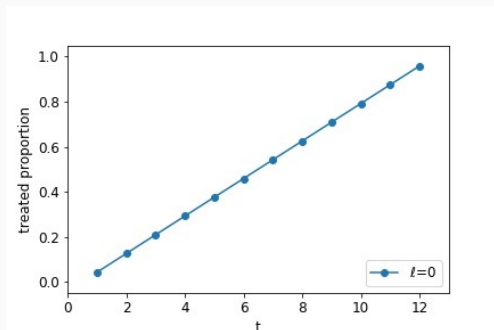
$$\text{Prec}(\hat{\tau}_0; Z) = \frac{Nt}{\sigma^2} \underbrace{(2\mathbf{b}_t^\top \boldsymbol{\omega}_{1:t} + \boldsymbol{\omega}_{1:t}^\top \mathbf{P}_{1:t} \boldsymbol{\omega}_{1:t})/t}_{g_\tau(\boldsymbol{\omega}, t)}$$

with

- $\boldsymbol{\omega}_{1:t} = [\omega_s]_{s \in [t]}$  and  $\omega_s = N^{-1} \sum_i Z_{is}$
  - $\mathbf{P}_{1:t} = I_t - \mathbf{1}_t \mathbf{1}_t^\top / t$  and  $\mathbf{b}_t$  is a vector of constants
  - $\sigma^2 = E[\varepsilon_{it}^2]$
- ) Termination rule needs key unknown parameter  $\sigma^2$
- ) Implement termination rule in a way that allows for valid inference of  $\tau_0$  (due to the peeking challenge in sequential testing (Johari et al. 2017))

## Decision 2: Treatment assignment

$\tilde{T}$  is unknown for adaptive experiments, therefore infeasible to optimally choose the speed of treatment rollout, **pre-experiment**



$$\omega_{0,s}^* = (2s - 1)/(2T)$$

# Three competing goals in adaptive experiments

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Goal 1: Choosing a treatment design

- Adaptively choose the speed of rollout, as we gather more information about  $\sigma^2$  during the experiment

Goal 2: Implementing the termination rule

- Estimate  $\sigma^2$  to make the next challenge manageable

Goal 3: Efficient estimation and valid inference for  $\tau_0$

- Use as many observations as possible

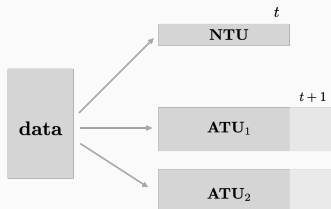
Propose the Precision-Guided Adaptive Experiment (PGAE) algorithm

- **simultaneously** achieves the **three** goals
- uses **sample splitting** and **dynamic programming**



Partition units into **non-adaptive** treatment units (NTU) and **adaptive** treatment units (ATU)

- **NTU**: Treatment design set pre-experiment (a small set)
  - Set as  $\omega_{bm,s} = (2s - 1)/(2T_{\max})$  (optimal solution for  $T_{\max}$ )
- **ATU**: Treatment design chosen adaptively



## Component 1 in PGAE: Choosing a treatment design

At time  $t$ , estimate **distribution of  $\sigma^2$**  from NTU

- Estimate  $\sigma^2 = E[\varepsilon_{it}^2]$  and variance of  $\varepsilon_{it}^2$ , i.e.,  $\xi^2 = E[(\varepsilon_{it}^2 - \sigma^2)^2]$
- **Normal approximation** of the distribution of  $\sigma^2$  (based on the asymptotic normality of  $\widehat{\sigma^2}$ )

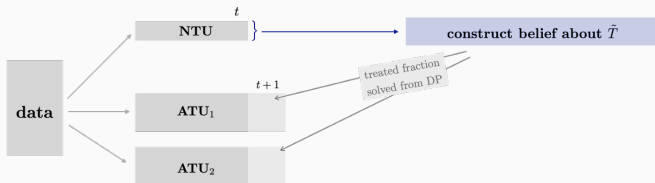
Update belief about  $\tilde{T}$ , denoted by  $P_t(\tilde{T})$ , using the estimated distribution of  $\sigma^2$



## Component 1 in PGAE: Choosing a treatment design

At time  $t$ , optimize  $\omega_{t+1}$  for ATU<sub>1</sub> and ATU<sub>2</sub> through dynamic programming (DP)

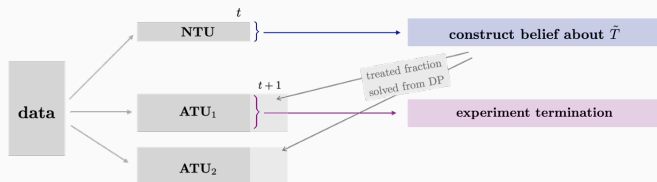
- In the DP, no intermediate cost and terminal cost is the precision at termination, i.e.,  $\text{Prec}(\hat{\tau}_0; Z_{:,1:\tilde{T}}) = (N\tilde{T}/\sigma^2) g_{\tilde{T}}(\omega, \tilde{T})$
- Solve  $\omega_{t+1}$  from DP based on the belief about  $\tilde{T}$



## Component 2 in PGAE: Implementing the termination rule

Estimate  $\sigma^2$  from  $ATU_1$  and  $\text{Prec}(\hat{\gamma}_0; Z_{:,1:t}) = (Nt/\hat{\sigma}^2) g_{\tau}(\omega, t)$

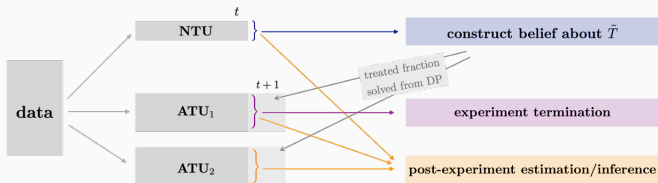
If  $\text{Prec}(\hat{\gamma}_0; Z_{:,1:t}) < c$ , terminate the experiment; otherwise, keep running the experiment



## Component 3 in PGAE: Efficient estimation and valid inference

Post-experiment,

- $\hat{\tau}_{\text{all}, \tilde{T}}$ : estimator of  $\tau_0$  using all  $N$  units and  $\tilde{T}$  periods of data (no efficiency loss)
- $\hat{\sigma}_{\text{atu}, 2, \tilde{T}}^2$ : estimator of  $\sigma^2$  using  $\tilde{T}$  periods of data of  $\text{ATU}_2$



## Theorem 2: Asymptotic distribution of estimators from PGAE

Suppose  $\varepsilon_{is}$  is bounded with a symmetric distribution around 0. As  $N \rightarrow \infty$ ,

$$\rho_{\overline{N}} \begin{bmatrix} (\tilde{T} g_{\tau}(\omega_{\text{all},1;\tilde{\tau}}, \tilde{T})/\sigma^2)^{1/2} (\hat{\tau}_{\text{all},\tilde{\tau}} - \tau_0) \\ (\tilde{T} \rho_{\text{atu},2}/\xi_{\tilde{\tau}}^{\dagger 2})^{1/2} (\widehat{\sigma}_{\text{atu},2,\tilde{\tau}}^2 - \sigma^2) \end{bmatrix} \overset{d}{\rightarrow} N(\mathbf{0}, I_2), \quad (3)$$

where  $\xi_{\tilde{\tau}}^{\dagger} = [\xi^2 + \sigma^4/(\tilde{T} - 1)]^{1/2}$  and  $\xi^2 = E[(\varepsilon_{it}^2 - \sigma^2)^2]$ .

- $\hat{\tau}_{\text{all},\tilde{\tau}}$  is consistent for  $\tau$  with the optimal convergence rate  $\rho_{\overline{N}}$ 
  - **Intuition:** Asymptotic conditional mean of  $\varepsilon_{is}$  on estimated even moments of  $\varepsilon_{is}$  is zero (due to the symmetric distribution of  $\varepsilon_{is}$ )
- $\widehat{\sigma}_{\text{atu},2,\tilde{\tau}}^2$  is consistent for  $\sigma^2$ 
  - **Intuition:** A different sample is used to estimate  $\widehat{\sigma}_{\text{atu},2,\tilde{\tau}}^2$

## Theorem 2: Asymptotic distribution of estimators from PGAE

Suppose  $\varepsilon_{it}$  is bounded with a symmetric distribution around 0. As  $N \rightarrow \infty$ ,

$$\rho_{\bar{N}} \begin{bmatrix} (\tilde{T} g_{\tau}(\omega_{\text{all},1;\tilde{T}}, \tilde{T})/\sigma^2)^{1/2} & (\hat{\tau}_{\text{all},\tilde{T}} - \tau_0) \\ (\tilde{T} \rho_{\text{atu},2}/\xi_{\tilde{T}}^{\dagger 2})^{1/2} & (\widehat{\sigma}_{\text{atu},2,\tilde{T}}^2 - \sigma^2) \end{bmatrix} \xrightarrow{d} N(\mathbf{0}, I_2), \quad (4)$$

where  $\xi_{\tilde{T}}^{\dagger} = [\xi^2 + \sigma^4/(\tilde{T} - 1)]^{1/2}$  and  $\xi^2 = E[(\varepsilon_{it}^2 - \sigma^2)^2]$ .

- The **adaptivity** of the design, with the **termination time** depending on early values of the outcomes, **comes at no cost** in the estimation of  $\tau_0$ 
  - Compare with a series of experiments with the same distribution of termination times, the average variance of  $\hat{\tau}_{\text{all},\tilde{T}}$  is the same
- Adaptive treatment decisions improve the estimation precision of  $\tau_0$ 
  - $g_{\tau}(\omega_{\text{all},1;\tilde{T}}, \tilde{T})$  is increased through adaptive treatment decisions

## Empirical application

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### MarketScan medical claims databases

- Inpatient and outpatient claim records from early 2007 to mid 2017
- Primary diagnosis is influenza 21,277 inpatient and 9,678,572 outpatient admissions

Study effect of interventions (e.g., face cover, social distancing, and vaccine) on flu occurrence rate

- Aggregate at the Metropolitan Statistical Area (MSA) level and month
- Focus on the flu peak season (October to April)

Other applications (medical home visits, grocery expenditure, and Lending Club loans) are in the paper

# Comparison of non-adaptive designs

## Benchmark designs

- $Z_{ff}$ : 50% control and 50% treated for all time periods
- $Z_{ba}$ : first half time periods all control, and second half **all** treated
- $Z_{ffba}$ : first half time periods all control, and second half **half** treated

## Non-adaptive staggered designs

- $Z_{opt}$ : nonlinear staggered design with  $\omega_s = \omega_{\ell,s}^*$
- $Z_{opt,linear}$ : linear staggered design with  $\omega_s = \omega_{0,s}^* = (2s - 1)/(2T)$
- $Z_{opt,stratified}$ : nonlinear staggered design with  $\omega_s = \omega_{\ell,s}^*$  and historical data used for stratification

0	0
0	0
1	1
1	1

$Z_{ff}$

0	1
0	1
0	1
0	1

$Z_{ba}$

0	0
0	0
0	1
0	1

$Z_{ffba}$

0	0
0	1
0	1
1	1

staggered designs

## Synthetic non-adaptive experimental data

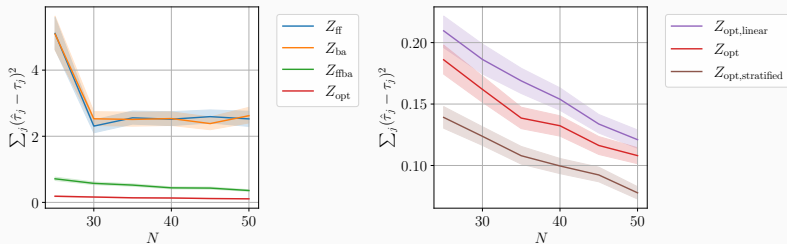
- Assume the synthetic treatment was not applied to the original data, so observed value =  $Y_{is}(\mathbf{0})$
- Apply a synthetic treatment using  $Z$  and obtain synthetic experimental data

$$Y_{is} = Y_{is}(\mathbf{0}) + \tau_0 Z_{is} + \tau_1 Z_{i,s-1} + \tau_2 Z_{i,s-2}$$

## Evaluation metrics

- Estimate  $\tau_0$ ,  $\tau_1$  and  $\tau_2$  from  $Y_{it}$ , and compare  $\sum_j (\hat{\tau}_j - \tau_j)^2$  from the data generated by various  $Z$
- Other evaluation metrics (estimation error of cumulative effects, recall and “precision”) in the paper

## Results for synthetic non-adaptive experiments



- $Z_{opt}$  requires fewer than 50% units to achieve the same estimation error as  $Z_{ff}$ ,  $Z_{ba}$ , and  $Z_{ffba}$
  - $Z_{opt,stratified}$  further saves at least 20% units to achieve the same estimation error as  $Z_{opt}$  and  $Z_{opt,linear}$
- ) Using our solution with historical data can substantially reduce the experimental cost

## Synthetic adaptive experimental data

- Run PGAE: The adaptive experiment is run for  $\tilde{T}$  periods with precision threshold  $c$
- Apply a synthetic treatment using  $Z$  and obtain synthetic experimental data

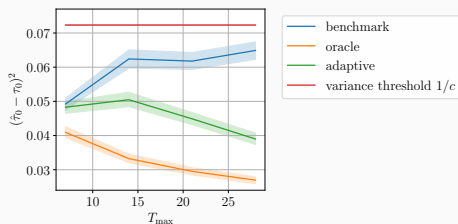
$$Y_{is} = Y_{is}(\mathbf{0}) + \tau_0 Z_{is}$$

## Three designs

- $Z_{adaptive}$ : design produced by PGAE with dimension  $N \times \tilde{T}$
- $Z_{benchmark}$ : design with  $\omega_s = (2s - 1)/(2T_{max})$  with dimension  $N \times \tilde{T}$  (optimal when  $\tilde{T} = T_{max}$ )
- $Z_{oracle}$ : design with  $\omega_s = (2s - 1)/(2\tilde{T})$  with dimension  $N \times \tilde{T}$  (assuming  $\tilde{T}$  is known ex-ante)

## Results for adaptive experiments

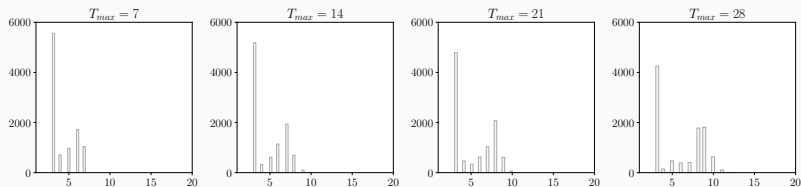
- Estimation error of the adaptive design always below variance threshold  $1/c$
- Adaptive design  $Z_{adaptive}$  reduces errors by 20% compared to benchmark design  $Z_{benchmark}$



## Termination time in adaptive experiments

For  $T_{\max} > 7$ , the experiment is always terminated quite early

) Desired precision threshold  $c$  achieved with less than  $T_{\max}/2$  duration



## Conclusion

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## Conclusion

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**Non-adaptive experiments:**  $N$ ,  $T$  and treatment decisions are determined, pre-experiment

- Analyze the statistical properties of GLS estimator of **instantaneous and lagged effects** from a **general outcome specification**
- Provide **analytical optimality conditions** that **maximize** a linear combination of **precisions** of estimated treatment effects
- Propose the treatment design that has **two** features: (1) **treatment fraction** takes an **S-shaped curve** in **time**; (2) **stratification**

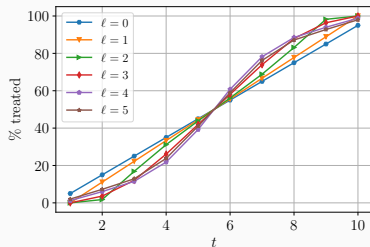
**Adaptive experiments:**  $N$  is fixed, and **experiment duration** and **treatment decisions** are determined **during the experiment**

- Propose the Precision-Guided Adaptive Experiment (**PGAE**) algorithm for **adaptive treatment design** and **post-experiment inference**
  - Combines ideas from **dynamic programming** and **sample splitting**
- Derive the **asymptotic normal distribution** of **final treatment effect** and **variance estimates** from PGAE
  - Final treatment effect estimate is **efficient** and achieves the **optimal convergence rate**

## **Supplementary material**

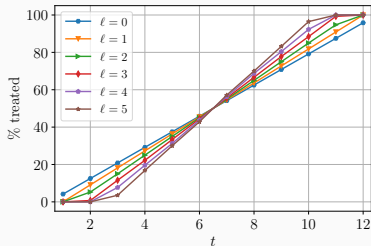
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## D-optimal design



D-optimal treatment design: Optimal treated proportion  $\omega_t$  at each period for a  $T$ -period treatment design and various  $\ell$ , where  $T = 10$ . Different colors represent different  $\ell$ .

## T-optimal design



T-optimal treatment design: Optimal treated proportion  $\omega_t$  at each period for a  $T$ -period treatment design and various  $\ell$ , where  $T = 12$ . Different colors represent different  $\ell$ .

## Expression of $\omega_{l,s}$

$$\omega_{l,s}^* = \begin{cases} 0 & s \leq bl/2c \\ a_{s-\lfloor l/2 \rfloor}^{(l)} & bl/2c < s \leq l \\ (2s - (l+1))/(2(T-l)) & l < s \leq T-l \\ 1 - \omega_{l,T+1-s}^* & T-l < s \leq T \\ 1 & T < bl/2c < s \end{cases} \quad (5)$$

## Expression of $\omega_{\ell,s}$

$a^{(\ell)}$  is defined as

$$a^{(\ell)} = (1 + (M^{(\ell)})^{-1} b^{(\ell)})/2,$$

where  $M^{(\ell)}$  and  $b^{(\ell)}$  are defined as

$$M^{(\ell)} = \begin{bmatrix} \lfloor \ell/2 \rfloor + 1 & & & & & \\ & \lfloor \ell/2 \rfloor + 2 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \ell & \end{bmatrix} - \frac{1}{T - \ell} \begin{bmatrix} \ell - \lfloor \ell/2 \rfloor & \ell - 1 - \lfloor \ell/2 \rfloor & \ell - 2 - \lfloor \ell/2 \rfloor & \cdots & 1 \\ \ell - 1 - \lfloor \ell/2 \rfloor & \ell - 1 - \lfloor \ell/2 \rfloor & \ell - 2 - \lfloor \ell/2 \rfloor & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

$$b^{(\ell)} = - \begin{bmatrix} \lfloor \ell/2 \rfloor + 1 \\ \vdots \\ \ell - 1 \\ \ell \end{bmatrix} + \frac{1}{T - \ell} \begin{bmatrix} (\lfloor \ell/2 \rfloor + 1)^2 \\ \vdots \\ (\ell - 1)^2 \\ \ell^2 \end{bmatrix} - \frac{1}{T - \ell} \begin{bmatrix} \sum_{l=1}^{\ell - \lfloor \ell/2 \rfloor} (\lfloor \ell/2 \rfloor + 1 - l) \\ \vdots \\ 2\lfloor \ell/2 \rfloor - 1 \\ \lfloor \ell/2 \rfloor \end{bmatrix}$$

## Examples of $\omega_{\ell,s}$

If  $\ell = 1$ , then

$$\omega_{\ell,s}^* = (s - 1)/(T - 1)$$

If  $\ell = 2$ , then

$$\omega_{\ell,1}^* = 0, \quad \omega_{\ell,2}^* = 1/(2T - 5)$$

$$\omega_{\ell,s}^* = (2t - 3)/2(T - 2) \text{ for } t = 4, \dots, T - 3,$$

$$\omega_{\ell,T-1}^* = 1 - 1/(2T - 5), \quad \omega_{\ell,T}^* = 1.$$

If  $\ell = 3$ , then

$$\omega_{\ell,1}^* = 0, \quad \omega_2^* = \frac{3}{6T^2 - 44T + 79}, \quad \omega_3^* = \frac{6(T - 4)}{6T^2 - 44T + 79},$$

$$\omega_t^* = \frac{t}{T} \frac{2}{3} \text{ for } t = 4, \dots, T - 3,$$

$$\omega_{T-2}^* = 1 - \frac{6(T - 4)}{6T^2 - 44T + 79}, \quad \omega_{T-1}^* = 1 - \frac{3}{6T^2 - 44T + 79}, \quad \omega_T^* = 1.$$

# An algorithm to choose a treatment design

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**Algorithm 1:** Choose a treatment design for each stratum  $g$

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```
1 Inputs:  $jO_{g,j}, [\omega_{\ell,t}^*]_{t \in [T]}$ 
2 for  $t = 1, \dots, T$  do
3    $N_{\text{treated},g,t}^{\text{int}} \leftarrow b j O_{g,j} \omega_{\ell,t}^* C;$ 
4    $N_{\text{treated},g,t}^{\text{dec}} \leftarrow j O_{g,j} \omega_{\ell,t}^* N_{\text{treated},g,t}^{\text{int}};$ 
5   if  $N_{\text{treated},g,t}^{\text{dec}} < 0.5$  or  $N_{\text{treated},g,t}^{\text{dec}} = 0.5$  with  $t < T/2$  then
6      $N_{g,t} \leftarrow N_{\text{treated},g,t}^{\text{int}};$ 
7   else
8      $N_{g,t} \leftarrow N_{\text{treated},g,t}^{\text{int}} + 1;$ 
9   end
10 end
11  $f(\cdot)$  a random function that shuffles  $1, 2, \dots, jO_{g,j}$ ;
12  $Z_g \leftarrow [0]^{O_g \times T};$ 
13 for  $i = 1, \dots, jO_{g,j}$  do
14   for  $t = 1, \dots, T$  do
15     if  $f(i) = N_{g,t}$  then
16        $z_{g,it} \leftarrow 1;$ 
17     else
18        $z_{g,it} \leftarrow 0;$ 
19     end
20   end
21 end
22 return  $Z_g;$ 
```

---



## Estimators in adaptive experiments

Three estimators are used in adaptive experiments

Suppose The estimators use the data of units in a set  $S$  over  $t$  periods collected so far, where  $t$  is small, but set size  $|S|$  can be large

### 1. within estimator for $\tau_0$

- Regresses  $\dot{Y}_{is}$  on  $\dot{Z}_{is}$  based on the specification  $\dot{Y}_{is} = \tau_0 \dot{Z}_{is} + \dot{\epsilon}_{is}$ , where for any variables  $f_{X_{is} | g_{(i,S)} \in S \times [t]}$  (e.g.,  $Y_{is}$  and  $Z_{is}$ ), and  $\dot{x}_{is}$  denotes the within transformed  $x_{is}$

$$\dot{x}_{is} = x_{is} - \bar{x}_i - \bar{x}_{\cdot S} + \bar{x},$$

in which  $\bar{x}_i$ ,  $\bar{x}_{\cdot S}$ , and  $\bar{x}$  are averages of  $x_{is}$ 's over  $t$  time periods, units in  $S$ , and both of them, respectively

## Estimators in adaptive experiments

### 2. Plug-in estimator for $\sigma^2$

$$\widehat{\sigma}_{S,t}^2 = \frac{1}{jSj} \frac{1}{(t-1)} \sum_{i \in S} \sum_{s=1}^t (\dot{y}_{is} - \hat{\tau}_{S,t} - \dot{z}_{is})^2$$

- The factor  $1/(t-1)$  is for finite  $t$  correction
- $\widehat{\sigma}_{S,t}^2$  is consistent and asymptotically normal for any finite  $t$

### 3. A new estimator for $\xi^2 = \mathbb{E}[(\varepsilon_{is}^2 - \sigma^2)^2]$

$$\widehat{\xi}_{S,t}^2 = \underbrace{\frac{t^2}{(t-1)^2}}_{\text{correction multiplier}} \underbrace{\frac{1}{jSj} \frac{1}{t} \sum_{i \in S} \left( \sum_{s=1}^t [(\dot{y}_{is} - \hat{\tau}_{S,t} - \dot{z}_{is})^2 - \widehat{\sigma}_{S,t}^2] \right)^2}_{\text{plug-in estimator of } \xi^2} \underbrace{\frac{3t-2}{(t-1)^2} (\widehat{\sigma}_{S,t}^2)^2}_{\text{correction term}}$$

- $\widehat{\xi}_{S,t}^2$  is consistent for any finite  $t$

## Asymptotic distribution for non-adaptive data

### Lemma: Asymptotic distribution of estimators from non-adaptive data

Suppose  $\varepsilon_{is}$  is i.i.d. for any  $i$  and  $s$  with  $E[\varepsilon_{is}] = 0$ ,  $E[\varepsilon_{is}^2] = \sigma_\varepsilon^2$ ,  $E[\varepsilon_{is}^3] = 0$ , and  $E[(\varepsilon_{is}^2 - \sigma^2)^2] = \xi^2$ .  $\hat{\tau}_{ntu,t}$  and  $\hat{\sigma}_{ntu,t}^2$  are consistent. As  $jS_{ntu} \rightarrow \infty$ , for any finite  $t$ , conditional on  $Z_{ntu}$ , we have

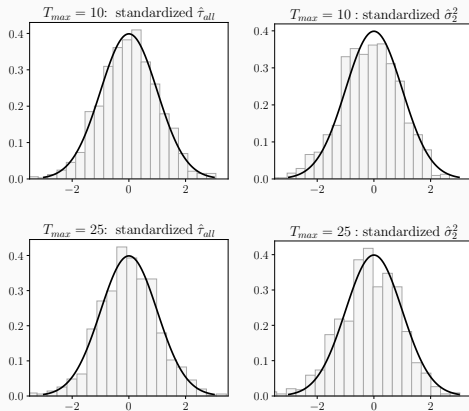
$$\sqrt{jS_{ntu}} \begin{pmatrix} \hat{\tau}_{ntu,t} \\ \hat{\sigma}_{ntu,t}^2 \end{pmatrix} \stackrel{d}{\rightarrow} \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2/(t-1) & g_\tau(\omega_{ntu,1:t}, t) \\ g_\tau(\omega_{ntu,1:t}, t) & 0 \end{bmatrix} \right),$$

where  $\xi_t^{\dagger 2} = \xi^2 + 2(\sigma^2)^2/(t-1)$ .

Furthermore,  $\sqrt{jS_{ntu}}(\hat{\xi}_t^2 - \xi^2) = O_p(1)$ .

) This lemma is used to prove Theorem 2

## Finite sample properties of Theorem 2



Finite sample properties of Theorem 2: Histograms of  $\hat{\tau}_{all,ss}$  and  $\hat{\sigma}_{atu,2,ss}^2$ . The standard normal density function is superimposed on the histograms.  $N = 500$ ,  $\tau_0 = 1$ , and  $\sigma_\varepsilon = 1$ .

## Within estimator for $\tau_0$

Least-squares estimator of  $\tau_0$  from the specification

$$Y_{is} = \alpha_i + \beta_s + \tau_0 Z_{is} + \varepsilon_{is}$$

is equivalent to the within estimator that regresses  $\dot{Y}_{is}$  on  $\dot{Z}_{is}$  based on the specification

$$\dot{Y}_{is} = \tau \dot{Z}_{is} + \dot{\varepsilon}_{is},$$

where for any variables  $\dot{X}_{is} \mathcal{G}_{(i,s) \in \mathcal{S} \times [t]}$  (e.g.,  $Y_{is}$  and  $Z_{is}$ ), and  $\dot{x}_{is}$  denotes the within transformed  $x_{is}$

$$\dot{x}_{is} = x_{is} - \bar{x}_{j\cdot} - \bar{x}_{\cdot s} + \bar{x},$$

in which  $\bar{x}_{j\cdot}$ ,  $\bar{x}_{\cdot s}$ , and  $\bar{x}$  are averages of  $x_{is}$ 's over  $t$  time periods, units in  $\mathcal{S}$ , and both of them, respectively

## Proof of Theorem 2: Key challenge

The estimation error of  $\hat{\tau}_{\text{all},t}(N)$  depends on  $\varepsilon_{is}$  (using data of  $N$  units and  $t$  periods)

$$\hat{\tau}_{\text{all},\bar{t}}(N) - \tau = \left( \sum_{i \in [N], s \leq \bar{t}} \dot{z}_{is}^2 \right)^{-1} \sum_{i \in [N], s \leq \bar{t}} \dot{z}_{is} \varepsilon_{is}.$$

The estimation error of the **plug-in estimator** for  $\sigma^2$  also depends on  $\varepsilon_{is}$

$$\begin{aligned} \widehat{\sigma}_{S,t}^2(N) &= \frac{1}{jSj(t-1)} \sum_{i \in S} \sum_{s=1}^t (\dot{y}_{is} - \hat{\tau}_{S,t} \dot{z}_{is})^2 \\ &= \frac{1}{jSj(t-1)} \sum_{i,s} \varepsilon_{is}^2 - \frac{t}{jSj(t-1)} \sum_i \bar{\varepsilon}_{i,\cdot}^2 - \frac{1}{t-1} \sum_s \bar{\varepsilon}_{\cdot,s}^2 + \frac{t}{t-1} \bar{\varepsilon}^2 \\ &\quad (\hat{\tau}_{S,t}(N) - \tau)^2 \frac{1}{jSj(t-1)} \sum_{i,s} \dot{z}_{is}^2 \end{aligned}$$

- **Key challenge:** We need to show  $\hat{\tau}_{\text{all},\bar{t}}(N)$  is “well-behaved” even if we condition on  $\widehat{\sigma}_{S,t}^2(N)$  that is used to make adaptive treatment decisions ( $S = \text{NTU}$ ) and experiment termination ( $S = \text{ATU}_2$ )

## Proof of Theorem 2: Two key properties

We leverage two critical properties

- **First property:** Given that  $\varepsilon_{is}$  has a symmetric distribution,  
 $E[\varepsilon_{is} j \widehat{\sigma}_{S,t}^2(N)] = 0$   
) The asymptotic mean of  $\widehat{\tau}_{\text{all}, \tilde{\tau}}(N)$  is zero
- **Second property:** Given that  $\widehat{\sigma}_{S,t}^2(N)$  is consistent,  
 $E[\varepsilon_{is}^2 \sigma^2 j \widehat{\sigma}_{S,t}^2(N)] = \widehat{\sigma}_{S,t}^2(N) \sigma^2$  converges to zero in probability  
) The asymptotic variance of  $(\check{T} g_{\tau}(\omega_{\text{all}, 1: \tilde{\tau}}, \check{T}) / \sigma^2)^{1/2} (\widehat{\tau}_{\text{all}, \tilde{\tau}} - \tau_0)$  is 1 (with probability approaching one, the variance is sufficiently close one)