DATASCI 385 Quantitative Finance

Lecture 5: Return and risk

Instructor: Ruoxuan Xiong

Suggested reading: Investments Ch 5

Lecture plan

- Risk and risk premiums

- Learning from historical data

Risk

- Any investment involves some uncertainty about future holding-period returns, and in many cases that uncertainty is considerable

- Sources of investment risk
  - macroeconomic fluctuations
  - changing fortunes of various industries
  - firm-specific unexpected developments

Holding-period returns

- The *holding-period return*, or HPR (rate of return over the entire time) equals

$$\text{HPR} = \frac{\text{Ending price} - \text{Beginning price} + \text{Cash dividend}}{\text{Beginning price}}$$

$$= \underbrace{\frac{\text{Ending price} - \text{Beginning price}}{\text{Beginning price}}}_{\text{rate of capital gains}} + \underbrace{\frac{\text{Cash dividend}}{\text{Beginning price}}}_{\text{dividend yield}}$$

- Suppose the initial cost of investment is $\$100$. The end-of-year price is $\$110$ and cash dividends over the year amount to $\$4$. Then HPR of a year is

$$\text{HPR} = \frac{110 - 100 + 4}{100} = 0.14$$

Probability distribution of the HPR

- There is considerable uncertainty about share prices one year from now, so you cannot be sure about your eventual HPR

	A	B	C	D	E	F	G	H	I
1									
2									
3	Purchase Price =		$100			T-Bill Rate =	0.04		
4									
5							Squared		Squared
6	State of the		Year-End	Cash		Deviations	Deviations	Excess	Deviations
7	Economy	Probability	Price	Dividends	HPR	from Mean	from Mean	Returns	from Mean
8	Boom	0.25	126.50	4.50	0.3100	0.2124	0.0451	0.2700	0.0451
9	Normal growth	0.45	110.00	4.00	0.1400	0.0424	0.0018	0.1000	0.0018
10	Mild recession	0.25	89.75	3.50	–0.0675	–0.1651	0.0273	–0.1075	0.0273
11	Severe recession	0.05	46.00	2.00	–0.5200	–0.6176	0.3815	–0.5600	0.3815
12	Expected Value (mean)			SUMPRODUCT(B8:B11, E8:E11) =	0.0976				
13	Variance of HPR					SUMPRODUCT(B8:B11, G8:G11) =	0.0380		
14	Standard Deviation of HPR					SQRT(G13) =	0.1949		
15	Risk Premium						SUMPRODUCT(B8:B11, H8:H11) =	0.0576	
16	Standard Deviation of Excess Return							SQRT(SUMPRODUCT(B8:B11, I8:I11)) =	0.1949

Mean of the HPR

- **Mean return**, $E(r)$: probability-weighted average of the rates of return in each scenario

$$E(r) = \sum_s p(s)r(s)$$

  - $p(s)$: the probability of scenario s
  - $r(s)$: the HPR in scenario s
  - $E(r) = (.25 \times .31) + (.45 \times .14) + [.25 \times (-.0675)] + [.05 \times (-.52)] = .0976$

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Variance of the HPR

- **Variance**, σ^2: the expected value of the *squared* deviation from the mean

$$Var(r) = \sigma^2 = \sum_s p(s)[r(s) - E(r)]^2$$

- $\sigma^2 = .25(.31 - .0976)^2{}_{2} + \ .45(.14\ -\ .0976)^2 + .25(-.0675\ -\ .0976)^2 + .05(-.5$
  $.0380$

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Excess return and risk premium

- Excess return: difference between HPR and risk-free rate r_f

$$r(s) - r_f$$

  - Excess return is random
  - Risk-free rate is nonrandom (T-bills rate)

- Risk premium: difference between expected HPR and risk-free rate r_f

$$E(r) - r_f$$

  - Risk premium is nonrandom

Sharpe ratio

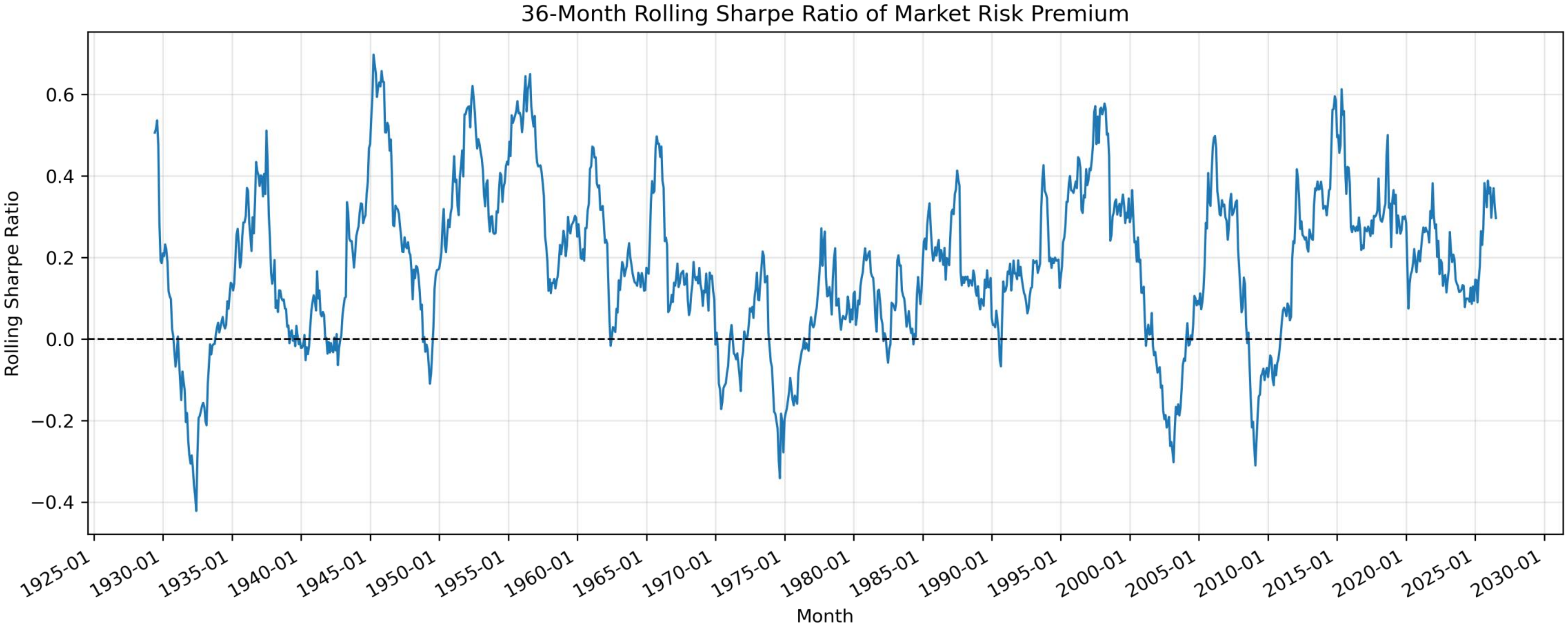
- Higher risk premium is always associated with higher risk (measured by SD)

- Sharpe ratio (reward-to-volatility ratio): risk premium divided by standard deviation

$$\text{Sharpe ratio} = \frac{\text{Risk premium}}{\text{SD of excess return}}$$

$$= \frac{E(r) - r_f}{\sigma}$$

Example: Sharpe ratio of market factor



Lecture plan

- Risk and risk premiums

- Learning from historical data

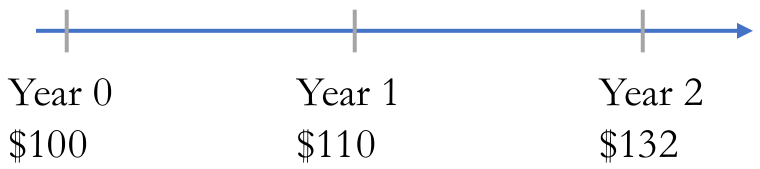
Expected returns and arithmetic average

- When we use historical data, we treat each observation as an equally likely “scenario”. If there are n observations,

$$E(r) = \sum_{s} p(s)r(s) = \frac{1}{n}\sum_{i=1}^{n} r_i$$

  - Arithmetic average of historic rates of return

  - Example
    - $r_1 = \frac{110}{100} - 1 = 0.1$
    - $r_2 = \frac{132}{110} - 1 = 0.2$
    - $E(r) = \frac{(0.1+0.2)}{2} = 0.15$



However, $100 \times (1 + 0.15)^2 = 132.25 \neq 132$

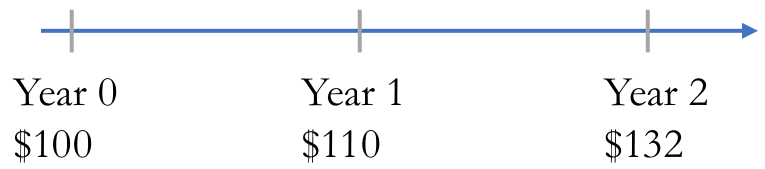
The geometric (time-weighted) average return

- *Geometric* or *compound* rate of return g: the fixed HPR that would compound to the same terminal value resulting from the sequence of actual returns in the time series (P_0: $Initial\ value$; P_n: $Terminal\ value$)

$$P_0 \times (1 + r_1) \times (1 + r_2) \times \ldots \times (1 + r_n) = P_n$$

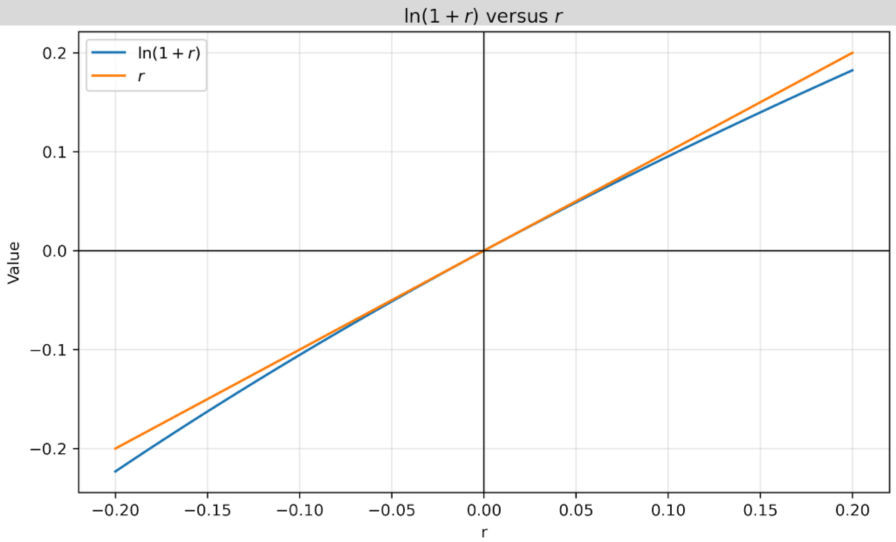
$$(1 + g)^n = \frac{P_n}{P_0}$$

  - *time-weighted* average return



  - Example
    - $(1 + g)^n = 132/100$
    - $g = \left(\frac{132}{100}\right)^{1/2} - 1 = 14.89\%$

Log return



- When r is small, $\ln(1+r) \approx r$
- Log return is $r_t = \ln\left(\frac{P_t}{P_{t-1}}\right)$
- Time additivity of r_t,

$$\ln(P_0) \times (r_1 + r_2 + \cdots + r_n) = \ln(P_n)$$

- Let average return be $\bar{r} = (r_1 + r_2 + \cdots + r_n)/n$. Then

$$n\bar{r} = \ln\left(\frac{P_n}{P_0}\right) \text{ or } P_0 e^{\bar{r}} \times e^{\bar{r}} \times \cdots \times e^{\bar{r}} = P_n$$

Estimating variance and standard deviation

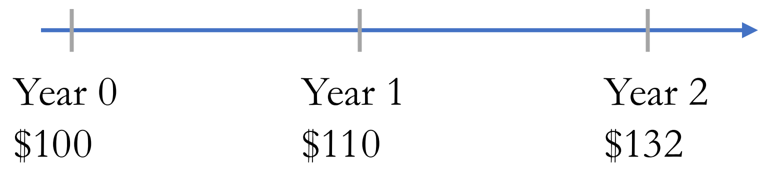
- We can estimate the variance of the actual returns from historical data

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{s=1}^{n}(r(s) - \bar{r})^2$$

- $\bar{r} = E(r) = \frac{1}{n}\sum_{s=1}^{n} r(s)$

- The denominator $n - 1$ is to account for the degree of freedom

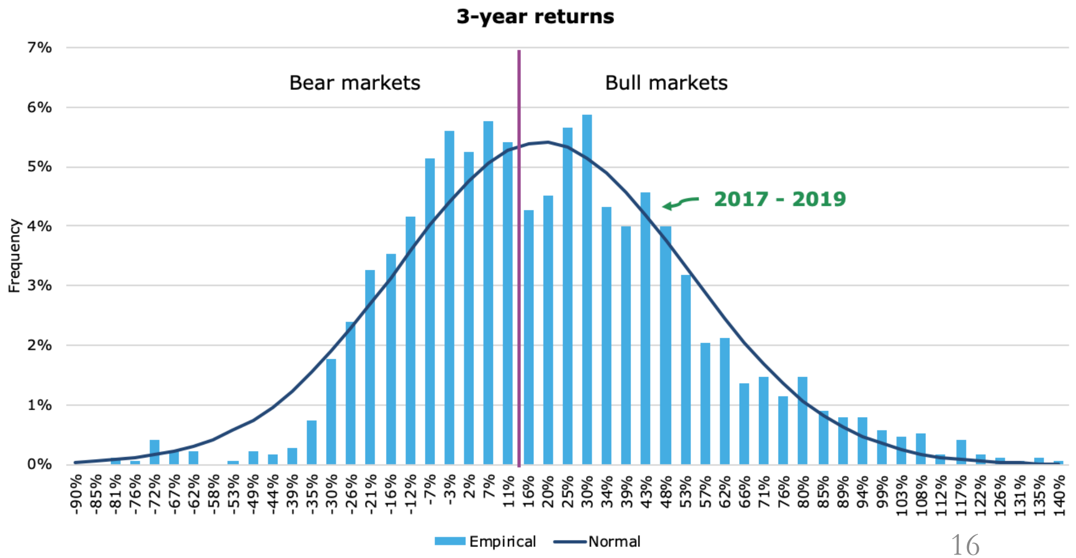
- Example
  - $\hat{\sigma}^2 = \frac{1}{2-1}[(0.2 - 0.15)^2 + (0.1 - 0.15)^2] = 0.005$
  - Standard deviation $\hat{\sigma} = 0.005^{1/2} = 0.071$



Normal approximation of distribution of returns

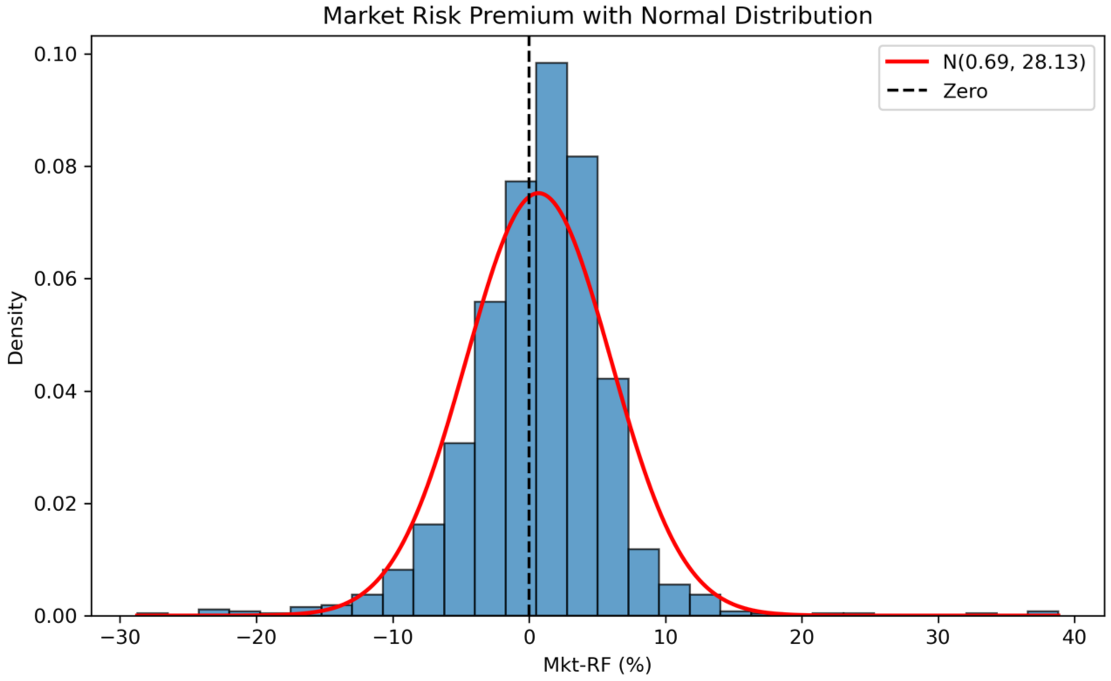
- Normal distribution appears naturally in many applications
  - E.g., the heights and weights of newborns, lifespans of many customers items
- Why does investment management use normal distribution?
  - Symmetricity: standard deviation is sufficient to capture the risk
  - Scenario analysis is simpler: only mean and variance are sufficient to estimate scenario probability
  - Easy to model statistical dependence of returns across assets: correlation is sufficient

- Let us look at the real data…



Deviation from normality

- First deviation: Asymmetry in the probability distribution of returns
- Second deviation: Likelihood of extreme values on either side of the mean



Measuring asymmetry

- Skew measures asymmetry in the probability distribution of returns

$$\text{Skew} = \text{Average}\left[\frac{(r-\bar{r})^3}{\hat{\sigma}^3}\right]$$

  - E.g., large negative returns are more likely than large positive returns
  - Negative skew: extreme *bad* outcomes are more frequent than extreme positive ones (skew to the left, fatter left tail, underestimate risk)
  - Positive skew: opposite case (skew to the right)

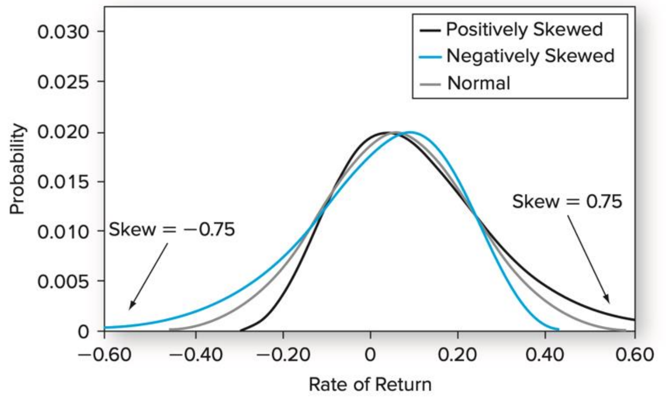


Figure 5.4 Normal and skewed distributions (mean = 6%, SD = 17%)

Measuring extreme values

- Kurtosis measures likelihood of extreme values on either side of the mean

$$\text{Kurtosis} = \text{Average}\left[\frac{(r-\bar{r})^4}{\hat{\sigma}^4}\right] - 3$$

  - Deviations are raised to the fourth power so more sensitive to extreme outcomes
  - Subtract by 3 because the kurtosis for the normal distribution is 3